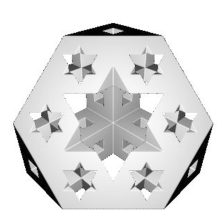
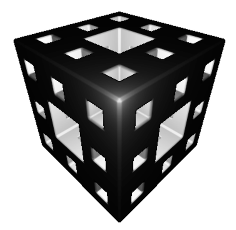
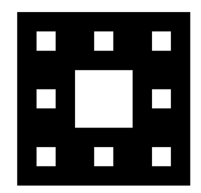
Menger Sponges | *Sierpiński Carpet* | *Cantor Set*



M_2 *Cross Section*



M_2



S_2

$C_0, \ldots\ldots, C_7.$

COMPREHENDING THE *MENGER SPONGE*

BASSIL ALCHEIKH, BARETT A. LESLIE, BRIAN PICHARDO

ILLINOIS INSTITUTE OF TECHNOLOGY

ABOUT KARL MENGER

Karl Menger

Image Source: Illinois Institute of Technology

Karl Menger was born in *Vienna, Austria*, on *January 13*, ***1902***, to the economist **Carl Menger** and **Hermione Andermann**. Throughout his childhood, **Menger** excelled at writing and the sciences, and looked forward to devoting his studies to physics by the time he enrolled at the *University of Vienna* in ***1920***. A lecture by Austrian mathematician **Hans Hahn** on *Neueres über den Kurvenbegriff (What's New Concerning the Concept of a Curve)*, however, inspired **Menger** to pursue a life-long, academic relationship with mathematics. **Hahn** mentioned that at that time no one had ever successfully given a satisfactory definition of a curve—to which **Menger**, within a week, produced a succinct and pleasing definition. Unfortunately, **Menger's** diagnosis with tuberculosis led him out of *Vienna*. During his time away, he developed his definition of a curve further and, in ***1922***, submitted his complete work. In ***1926***, **Menger** discovered the fractal curve currently known as the "*Menger sponge*," during his research on dimensional theory. He was later offered a job at the *University of Notre Dame*, and thus immigrated to the *United States*. After his time at *Notre Dame*, he moved to *Chicago* and became a professor at the *Illinois Institute of Technology* in ***1946***. During his time at *IIT*, he wrote and published a calculus textbook that would improve the way college calculus was taught all over the country. He was a well-respected and beloved member of the *IIT* faculty, and lived in *Chicago* until his death on *October 5*, ***1985***.

1 AND 2 DIMENSIONAL ANALOGOUS SETS

Cantor Set

The *Cantor Set*, named after **Georg Cantor**, is a ternary set whose middle thirds are omitted at all iterations. The formal definition of the *Cantor Ternary Set* is:

$$\{\sum_{i=1}^{\infty}\frac{a_i}{3^i}, where\ a_i \in \{0,2\}.\}$$

note: $\sum_{i=1}^{\infty}\frac{a_i}{3^i}$, can be considered the ternary representation $\{0.a_1, a_2, a_3 \ldots a_i\}$, similary in the following cases: Sierpiński Carpet, & Menger Sponge.

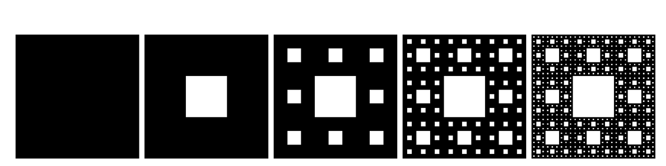
The graphic below depicts the first 7 iterations of the Cantor Set: $C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7$.

Sierpiński Carpet

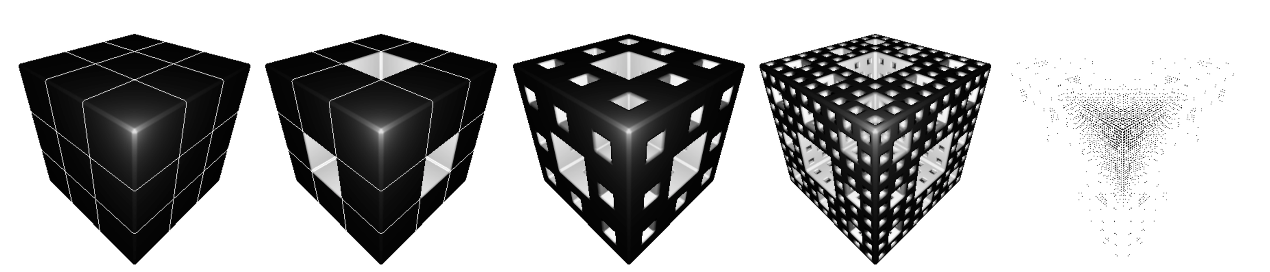
The *Sierpiński Carpet*, named after **Wacław Franciszek Sierpiński**, is the two-dimensional extension of the *Cantor Set*. It is defined as follows:

$$\{\left(\sum_{i=1}^{\infty}\frac{a_i}{3^i}, \sum_{j=1}^{\infty}\frac{b_j}{3^j}\right), where\ a_i, b_j \in \{0,1,2\},$$
$$for\ all\ i, j, but\ for\ any\ particular\ j\ at\ most\ one\ of\ a_j, b_j\ can\ equal\ 1.\}$$

The graphic below depicts the first 4 iterations of the *Sierpiński Carpet*: S_0, S_1, S_2, S_3, S_4.



CONSTRUCTING THE MENGER SPONGE



M_0 M_1 M_2 M_3 M_n

1. Begin with a unit cube.
2. Divide every face of the cube into 9 sub-squares. Giving 27 smaller cubes, (***M_0***).
3. Remove the smaller cube in the middle of each face, and remove the smaller cube in the very center of the larger cube, leaving 20 smaller cubes. This gives the first iteration, (***M_1***).
4. Repeat steps 2 and 3 for each of the remaining smaller cubes, and continue to iterate ad infinitum, (***M_n***).

THE MENGER SPONGE DEFINITION

The *Menger Sponge* is a three-dimensional extension of the *Cantor Set*. It is defined as follows:

$$\{\left(\sum_{i=1}^{\infty}\frac{a_i}{3^i}, \sum_{j=1}^{\infty}\frac{b_j}{3^j}, \sum_{k=1}^{\infty}\frac{c_k}{3^k}\right), where\ a_i, b_j, c_k \in \{0,1,2\}\ for\ all\ i, j, k\ but\ for\ any\ particular\ j\ at\ most\ one\ of\ a_j, b_j, c_j\ can\ equal\ 1.\}$$

This definition produces decimal (base 10) coordinates that give points within the boundaries of the *Menger Sponge* in 3D. The *Menger Sponge* can also be represented in ternary (base 3) as:

$$\{\{0.a_1, a_2, a_3 \ldots a_i, 0.b_1, b_2, b_3 \ldots b_j, 0.c_1, c_2, c_3 \ldots c_k\}_3, where\ a_i, b_j, c_k \in \{0,1,2\}\ for\ all\ i, j, k\ but\ for\ any\ particular\ j\ at\ most\ one\ of\ a_j, b_j, c_j\ can\ equal\ 1.\}$$

One intresting property of the Menger Sponge is that the Surface Area goes to ∞ *& Volume goes to* 0 *for the* n^{th} *iterate,* M_n.

VOLUME

At each iteration, each cube of the *Menger Sponge* is broken into 27 smaller cubes; of those 27, 7 of them are omitted. Thus, the number of cubes in any iteration of the *Menger Sponge* is: $N_n = 20^n$, where n is the number of iterations. Let L_n be the length of the side being removed and V_0 be the volume of the initial cube, which is constant. The Volume of the *Menger Sponge* at any iteration n is:

$$L_n = \left(\frac{1}{3}\right)^n = 3^{-n}, \quad \therefore V_n = L_n^3 N_n = \left(\frac{20}{27}\right)^n$$

Notice: if one takes the limit of V_n to infinity, V_n converges to 0. In other words, the *Menger Sponge's* volume should equal zero, as fractals are defined to have infinite iterations.

SURFACE AREA

The surface area of the *Menger Sponge* is not as straightforward to calculate as the volume. After using the computer software *Mathematica* to generate several values for the *Sponge's* surface area*, one can conjecture a surface area formula as a function of the *Sponge's* iteration:

$$A_n = \frac{2(20^n) + 4(8^n)}{9^n}$$

, where we assume an initial unit cube.

SURFACE AREA Cont…

To prove that this is formula works for any iteration, not merely the several iterations that *Mathematica* verified, one can use mathematical induction:

1. Base step (the zero*th* iteration, also known as the unit cube):

$$A_0 = \frac{2(20^0) + 4(8^0)}{9^0}$$
$$A_0 = \frac{2(1) + 4(1)}{1}$$
$$A_0 = 6$$

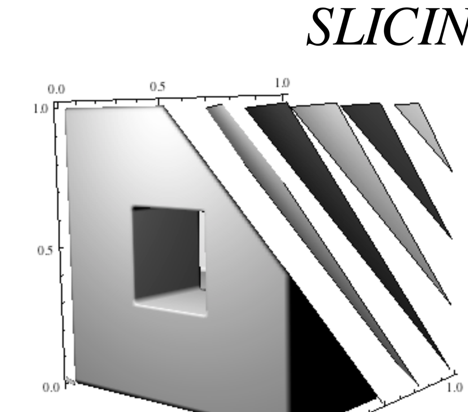
This can be verified by simply counting the number of faces on the unit cube (6), multiplied by the surface area on each cube (1×1=1).

2. Inductive Step:

$$A_{n+1} = \frac{20 \cdot 9^n (A_n) - 8^n (48)}{9^{n+1}}$$
$$A_{n+1} = \frac{20 \cdot \cancel{9^n}\left(\frac{2(20^n) + 4(8^n)}{\cancel{9^n}}\right) - 8^n (48)}{9^{n+1}}$$
$$= \frac{20 \cdot \left(2(20^n) + 4(8^n)\right) - 8^n (48)}{9^{n+1}}$$
$$= \frac{2(20^{n+1}) + 80 \cdot (8^n) - 8^n (48)}{9^{n+1}}$$
$$= \frac{2 \cdot 20^{n+1} + (8 \cdot 10)(8^n) - 8^n (6 \cdot 8)}{9^{n+1}}$$
$$= \frac{2 \cdot 20^{n+1} + 10 \cdot 8^{n+1} - 6 \cdot 8^{n+1}}{9^{n+1}}$$
$$= \frac{2 \cdot 20^{n+1} + (10-6) 8^{n+1}}{9^{n+1}}$$
$$A_{n+1} = \frac{2 \cdot 20^{n+1} + 4 \cdot 8^{n+1}}{9^{n+1}}$$

* The *Mathematica* code for the surface area:
Manipulate[(2 20^n + 4 8^n)/9^n, {n, 0, 4, 1.0}]

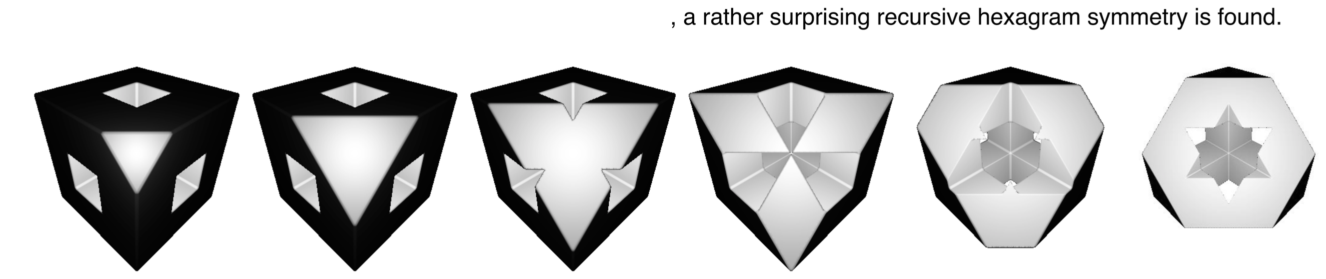
SLICING THE MENGER SPONGE



M_1 *Cross Sections*

$(x - n \times \frac{1}{12}) + (y - n \times \frac{1}{12}) + (z - n \times \frac{1}{12}) = 0$ is a plane which corresponds to the perpendicular bisector along the long horizontal of the *Menger Sponge*.

Using the computer algebra software, *Mathematica*, it is possible to slice a *Menger Sponge* along the plane. Giving two pieces so one may see the cross-section. By varying *n* in steps of 1, from 6 to 11, it is possible to show the various shapes which lie on or underneath the plane. At the plane: $(x - 6 \times \frac{1}{12}) + (y - 6 \times \frac{1}{12}) + (z - 6 \times \frac{1}{12}) = 0$, a rather surprising recursive hexagram symmetry is found.



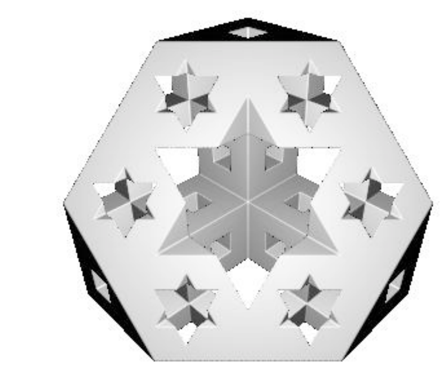
$n=11$ $n=10$ $n=9$ $n=8$ $n=7$ $n=6$

HERE n NOTATES THE EQUATION OF THE PLANE.

ABOUT THE CROSS SECTION

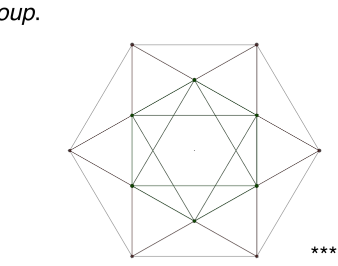
** HEXAGONAL SYMMETRY **

The resulting cross section reveals a very familiar hexagonal symmetry that has appeared throughout history. The hexagram has deep cultural, mathematical, and physical roots. The symmetry first appeared in ancient cultures and mythologies. Using Muslim sources, the English scholar and scientist, **Thomas Bradwardine**, first systematically studied *regular star polygons*, including the hexagram. One of the symmetry's most notable appearances is in *Group Theory*'s, ***G_2*** *Lie Group*.



M_2 *Cross Section*

G_2 as a subgroup of E_8 projected into the Coxter plane.



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CONCLUSIONS AND FUTURE WORK

The *Cantor Ternary Set*, *Sierpiński Carpet*, and *Menger Sponge* are analogous derived sets in 1, 2, and 3 dimensions, respectively. Through basic explorations of length, surface area, and volume, it is possible to see both the limitations and breadth of the *Cantor Set*, *Sierpiński Carpet* and *Menger Sponge*. Although the governing rules of these mathematical sets are simple at their essence, their resulting abstract concepts are uniquely powerful and have laid the groundwork for our understanding of dimension and curve.

FUTURE WORK

The hexagonal symmetry associated with the *Menger cross sections* requires a good explanation. We hope to further this research by finding similar iterative properties using the current derived sets or the possible exploration of others. Doing so will require us to refine our current techniques, algorithm development, and computer programing. Our current working hypothesis is that opposing iterations of the *Sierpiński Triangle* and *Sierpiński Tetrahedron* point to a symmetry that may appear at various iterations of the *Sierpiński Carpet* and *Menger Sponge*, respectively. By taking cross-sections of the higher dimensional *Menger Sponge*, it may be possible to find correlating 4*th* dimensional hexagon symmetry. Currently we are exploring the fourth dimensional *Menger Sponge* ternary notation.

ACKNOWLEDGEMENTS & REFERENCES

We would like to thank the following people for their assistance with the research:

Professor Greg Fasshauer (*guidance of this project*), Computational Mathematics Graduate Student Allen Flavell (*assisting in computer coding*), Craig Johnson (*3D modeling*), Professor Arthur Lubin (*ternary notation of the sets*), Erin Skvorc (*AutoCAD*), and Professor Fred Weening (*help and guidance of this project*).

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"Star Polygons", last accessed January 30, 2012. Weisstein, Eric W. "Star Polygon."
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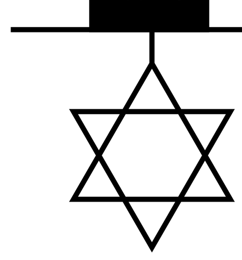
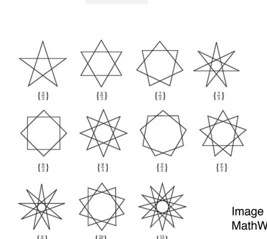
1700 BCE



Religious & Cultural Origins
Hinduism, Judaism, Ancient Greeks

1300



Thomas Bradwardine (1290 - 1349)
Star polygons

Image Source: MathWorld (http://mathworld.wolfram.com/StarPolygon.html)

1600

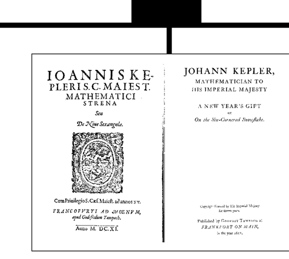


Image Source: Caltech (www.snowcrystals.com)

Johannes Kepler (1571 - 1630)
On the Six-Cornered Snowflake

1872

Georg Cantor (1845 - 1918)
Cantor Set

1901-1905

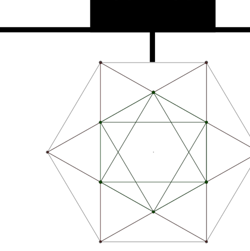
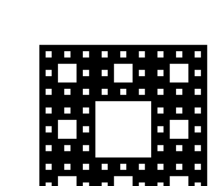


Image Source: VisibLie_E8 ***

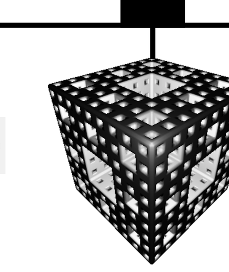
Leonard Eugene Dickson (1874 - 1954)
Mathematics finite groups G_2

1916



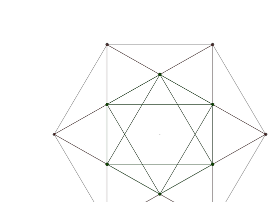
Wacław Sierpiński (1882 - 1969)
Sierpiński Carpet

1926



Karl Menger (1902 - 1985)
Menger Sponge

1932



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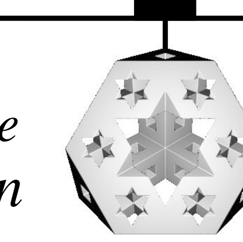
Werner Heisenberg (1901 - 1976)
Physics isospin

2011

Menger Sponge
Cross Section



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