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THE ENGINEERING SIGNIFICANCE OF LIMIT ANALYSIS

by

PHILIP G. HODGE, JR.

Research

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DEPARTMENT OF MECHANICS

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ABSTRACT

By definition, the limit load on a structure is the unique magnitude of the given loads under which a structure can first deform if it is made of a rigid-perfectly plastic material. The significance of the limit load for a structure made of a real material is discussed in relation to a simple truss.

The theory of limit analysis is based upon the idealization known as a perfectly plastic material. Such a material has a sharply defined yield stress under which the strains can increase indefinitely. In particular, if the material is rigid-perfectly plastic, it is assumed that no straining is possible for stresses below the yield stress. Let a given structure be under the action of a set of forces determined to within a magnitude factor P. As P is slowly increased, the unique value of P for which the rigid-perfectly plastic structure can first undergo any deformation is variously known as the limit load, yield point load, or collapse load.

If the structure is made of an elastic-perfectly plastic material, it can strain elastically for stresses below the yield stress and can never support any greater stress. For such a material, the yield point load is defined as that value of P for which indefinitely large deformations could occur if geometry changes could be neglected. Evidently this definition is somewhat more artificial. However, it can be shown[1] that the limit loads are the same for the rigid-perfectly plastic structure and the elastic-perfectly plastic structure.

Most real materials undergo a certain amount of strain hardening and can, in fact, support stresses greater than the yield stress. The purpose of the present note is to discuss the significance of the limit load as applied to such a material. Obviously, if the transition from elastic to plastic behavior is gradual, it will not be possible to give a sharp definition of the limit load. However, the limit load can be roughly defined as that value of the load above which small increases in load produce relatively larger increases in the displacements.

Figure 1 shows experimental stress strain data for two structurally important materials, mild steel[2] and 24S-T aluminum[3]. Both materials have quite sharply defined stress levels at which the behavior deviates from the linear. In the case of aluminum, strain hardening begins immediately; for mild steel there is a substantial increase in the strain before strain hardening. In each case, the elastic behavior is given by Hooke's law

$$\sigma = E\epsilon \qquad (1)$$

and the strain hardening portion is approximated by a power function

$$\sigma = B \epsilon^{n} \tag{2}$$

Table 1 shows the material constants associated with the two materials.

The symmetric three bar truss shown in Fig. 2 has been analyzed for the two real materials and their corresponding elastic-perfectly plastic materials. This truss has frequently been used in the literature,[4,5] as providing a simple example which nevertheless contains many essential characteristics of more complex structures. The details of the analysis are quite straightforward and are given in an appendix. The resulting curves of load vs. vertical displacement are shown in Figs. 3 and 4.

The applicability of the suggested definition of limit load will be discussed first in relation to the aluminum truss. In the fully elastic range, an increase of 10%, say, of the load is accompanied by an increase of 10% in the elongation. At a load of 76,700 pounds the central bar of the truss becomes plastic. To obtain a 10% increase in load to 84,400 pounds, the displacement must increase from 0.043 in. to 0.054 in., or 28%. Thus the ratio of displacement increase to load increase almost triples when the truss becomes partly plastic.

For a load of 107,900 pounds, the remaining bars become plastic. The displacement at this point is 0.082 in. A 10% increase in the load now brings it to 118,700 pounds. It follows from Fig. 3 that the displacement increases to 0.164 in., or by 100%. Thus the displacement rate - load rate ratio which less than tripled in the partly plastic range now more than triples again, and becomes ten times its elastic value.

For the steel truss, the results are even more striking. In the range of contained plastic deformation a 10% increase in load requires a 25% increase in displacement. At the instant the P first reaches the limit load, 89,000 pounds, the displacement is 0.0324 in; a 10% increase in load requires a displacement of 0.268 in., or an increase of 827%! Even if this latter displacement is compared with the displacement at the onset of hardening the increase is over 114%.

In conclusion, the limit load which has a precise meaning for the perfectly plastic material, has been shown to have qualitative significance for more realistic materials. Below the limit load, the deformation increase associated with a given load increment is of the same order of magnitude as in elasticity, whereas above the limit load it is greater by a factor of ten or more. It follows that if the load in a structure is restricted to values less than the limit load the deformations will remain on the elastic order, whereas above this load substantially larger deformations must be expected.

		Mild steel	Aluminum
Young's modulus	E	27.2×10^6 psi	12.51×10^6 psi
Yield stress	σ^*	36.8×10^3 psi	45.0×10^3 psi
Strain at initial hardening	ϵ_o	0.0117 in./in.	---
Hardening modulus	B	125.4×10^3 psi	94.65×10^3 psi
Hardening coefficient	n	0.275	0.136

Table 1

Physical characteristics of mild steel and aluminum

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APPENDIX

Horizontal equilibrium of the truss in Fig. 2 is satisfied by symmetry. Vertical equilibrium requires that

$$F_1 + \sqrt{2}\,F_2 = P \qquad (3)$$

where F_1 and F_2 are the forces in the bars. If L is the vertical displacement of the load, the extensions L_1 and L_2 of the bars are

$$L_1 = L, \quad L_2 = L/\sqrt{2} \qquad (4)$$

The forces and elongations are related to the stresses and strains by

$$F_1 = A_1\sigma_1 \quad , \quad F_2 = A_2\sigma_2$$
$$L_1 = H\epsilon_1 \quad , \quad L_2 = \sqrt{2}\,H\epsilon_2 \qquad (5)$$

where A is the cross sectional area.

In the elastic range, it follows from Eqs. (1), (4), and (5) that the forces and elongations are related by

$$F_1 = (EA_1/H)L \qquad (6)$$

$$F_2 = (EA_2/2H)L \qquad (7)$$

In the perfectly plastic range, the elongation is not directly determinate, and the forces are

$$F_1 = A_1\sigma^* \qquad (8)$$

$$F_2 = A_2\sigma^* \qquad (9)$$

Finally, in the strain hardening range it follows from Eqs. (2), (4), and (5) that

$$F_1 = A_1B(L/H)^n \qquad (10)$$

$$F_2 = A_2B(L/2H)^n \qquad (11)$$

We consider a truss with

$$A_1 = A_2 = 1 \text{ in.}^2 \quad , \quad H = 12 \text{ in.} \qquad (12)$$

For definiteness, we consider the case of mild steel; the aluminum truss is handled in a similar manner, and is, in fact, somewhat simpler.

For P sufficiently small, both bars are elastic. It follows from Eqs. (6), (7), and (3) that

$$P = 1.707\ (EL/H) = 3.87 \times 10^6\ L \qquad (13)$$
$$F_1 = 0.586\ P \quad , \quad F_2 = 0.293\ P$$

This solution remains valid until $F_1 = A_1\sigma^* = \sigma^*$, or

$$P = P^* = 62{,}700 \qquad (14)$$

For P somewhat greater than P^*, the vertical bar is perfectly plastic and the diagonal bars are elastic, hence Eqs. (8), (7), and (3) are to be used. Thus

$$P = \sigma^* + 0.707\ EL/H = 36{,}800 + 1.60 \times 10^6\ L \qquad (15)$$
$$F_1 = \sigma^* \quad , \quad F_2 = (P - \sigma^*)/\sqrt{2}$$

This solution remains valid until either $F_2 = A_2\sigma^*$ or $\epsilon_1 = \epsilon_0$. The former contingency occurs first for

$$P = P^{**} = 89{,}000 \qquad (16)$$

For $P = P^{**}$ the strains will increase until $\epsilon_1 = \epsilon_0$, i.e., until

$$L = L_3 = H\epsilon_0 = 0.140 \qquad (17)$$

The load may then increase again. The vertical bar is strain hardening so that Eqs. (10), (8), and (3) are applicable. This leads to

$$P = 63{,}300\ L^{0.275} + 52{,}000 \qquad (18)$$

When $\epsilon_2 = \epsilon_0$, the displacement is

$$L = L_4 = 0.280 \qquad (19)$$

and the corresponding value of the load is

$$P = P_4 = 96{,}400 \qquad (20)$$

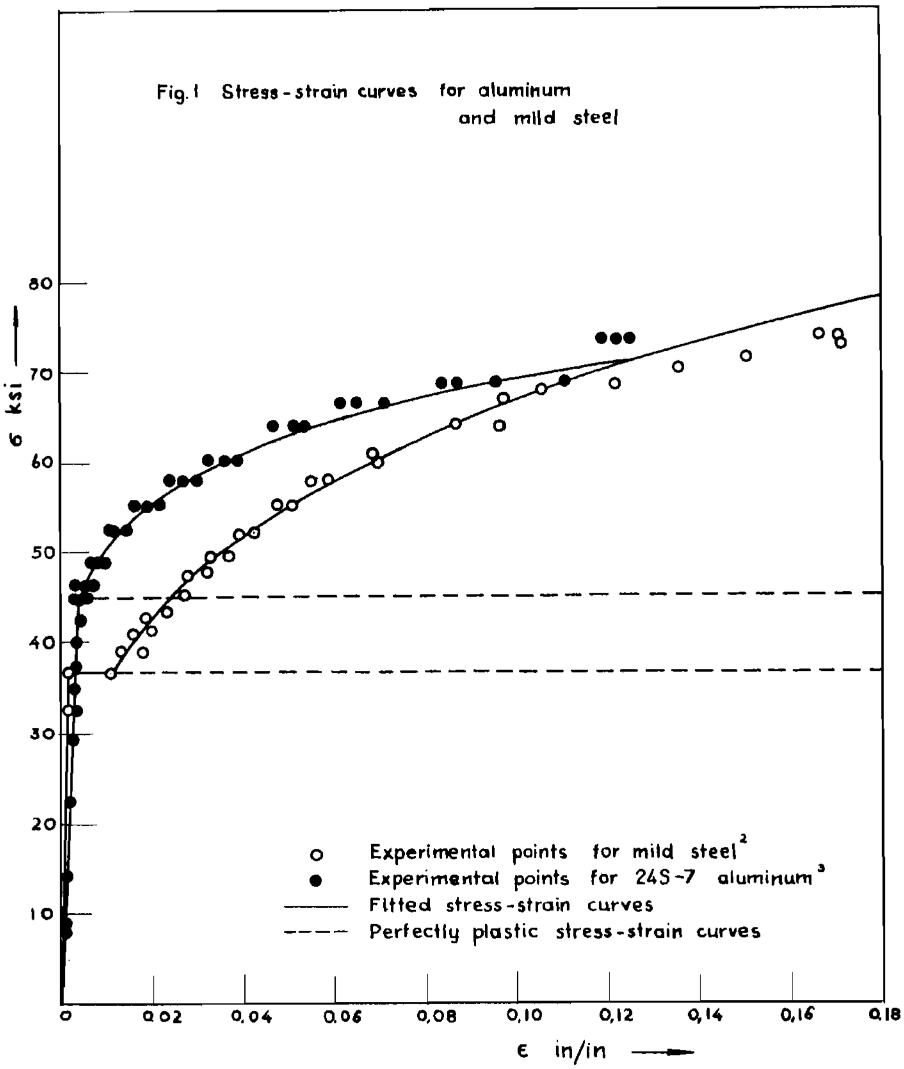
Finally, for $P > P_4$ both bars strain harden. Eqs. (10), (11), and (3) then lead to

$$P = 137{,}800\ L^{0.275} \qquad (21)$$

The complete relation between P and L is shown in Fig. 4.

The similar equations for the aluminum truss are readily verified to be

Elastic:	$P = 1.78 \times 10^6 L$	(22)
	$P^* = 76,700$	(23)
Partly plastic:	$P = 66,200\ L^{0.136} + 737,000L$	(24)
	$P^{**} = 107,900$	(25)
Fully plastic:	$P = 151,500\ L^{0.136}$	(26)



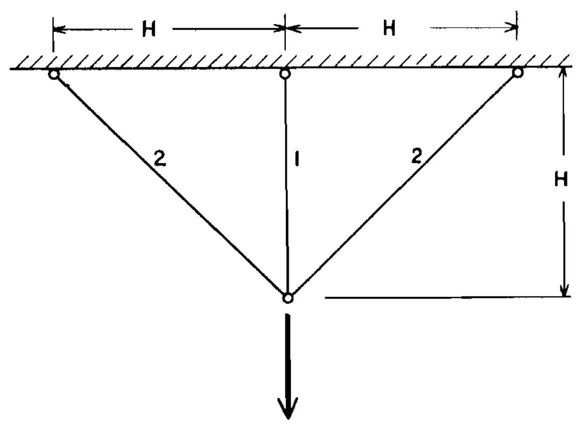


Fig. 2 Symmetric truss

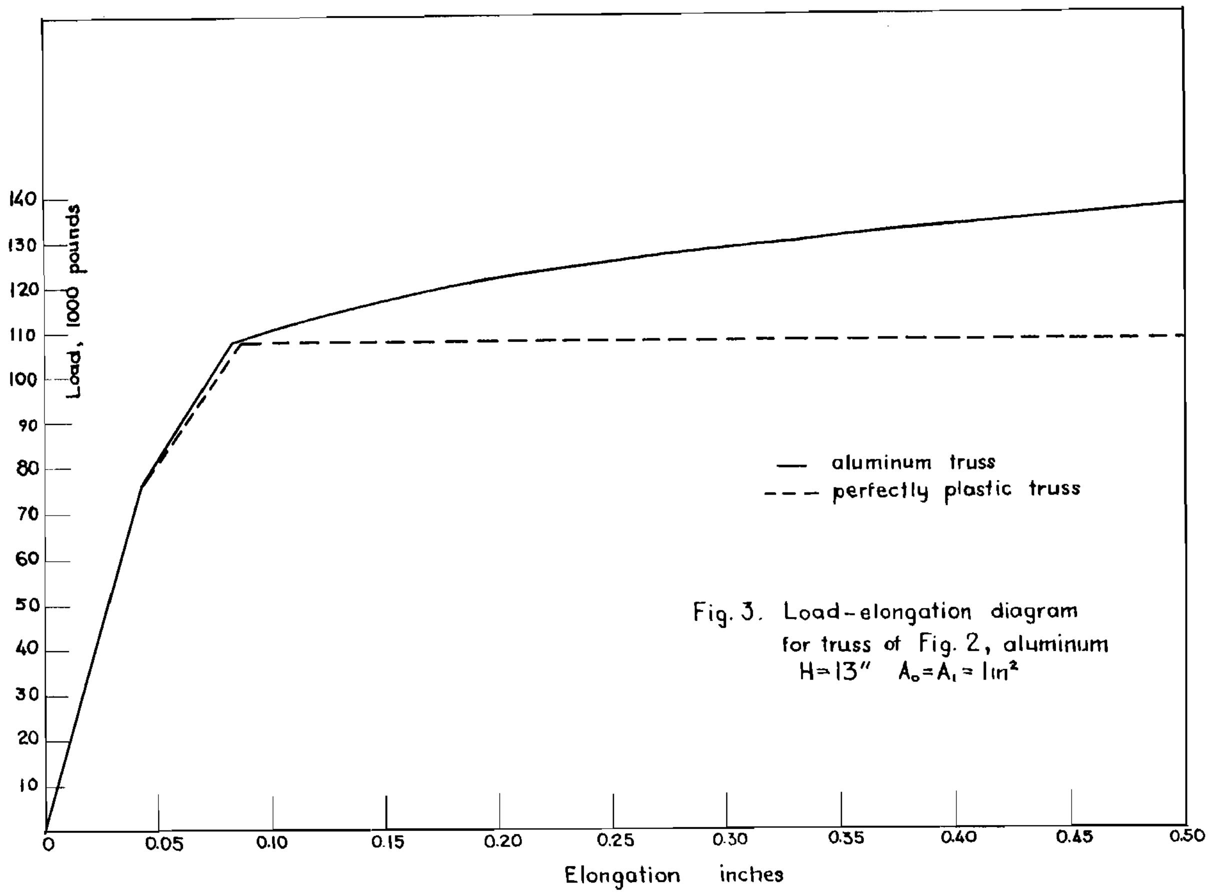


Fig. 3. Load-elongation diagram for truss of Fig. 2, aluminum
$H = 13''$ $A_o = A_1 = 1 in^2$

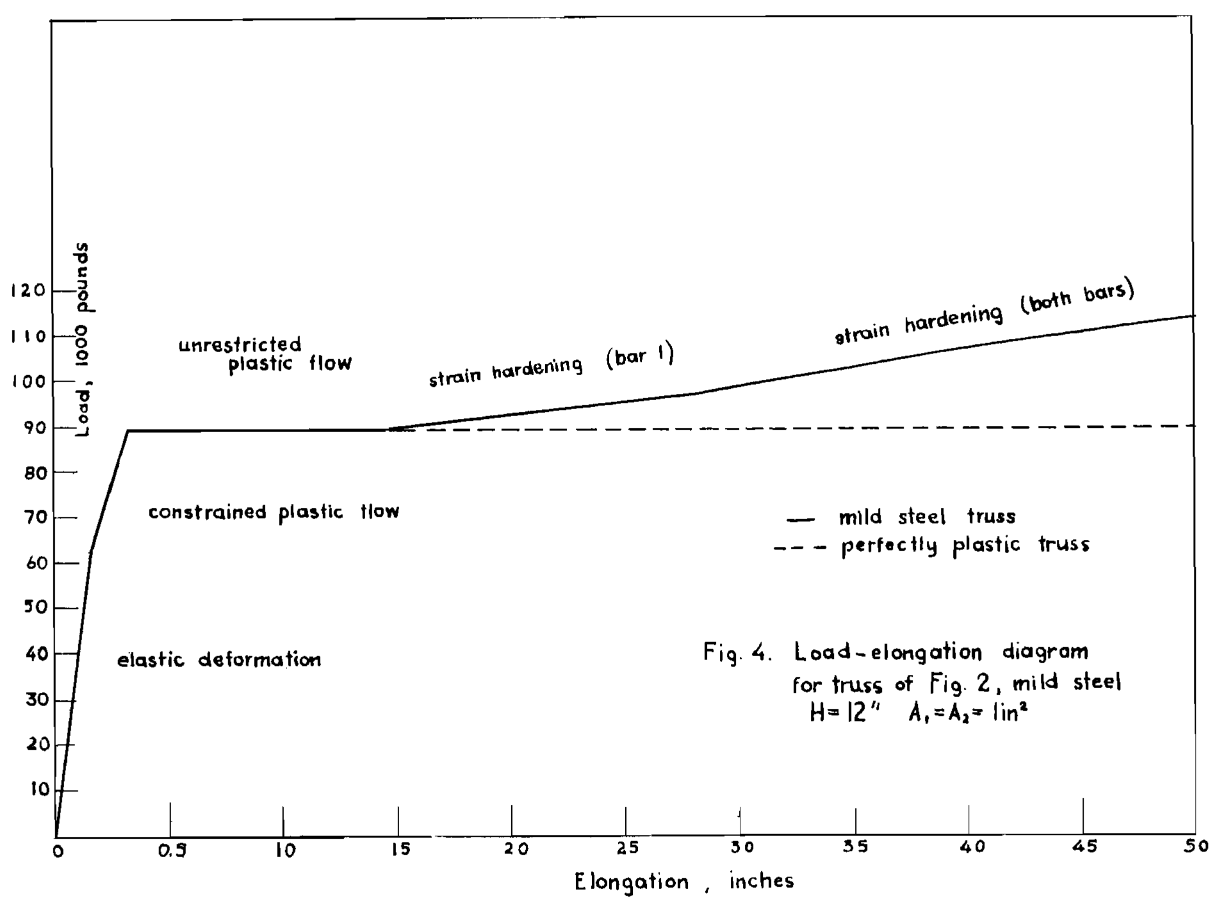


Fig. 4. Load-elongation diagram for truss of Fig. 2, mild steel
$H = 12''$ $A_1 = A_2 = 1 in^2$