High-order cumulants from the 3D $\boldsymbol{O(1)}$ and $\boldsymbol{O(4)}$ spin models

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We simulate the 3D $O(1)$ (Ising) and $O(4)$ spin models by the Monte Carlo method. Interesting high-order cumulants from the 3D Ising and $O(4)$ universality classes are presented and discussed. They all show the non-monotonic or sign change behavior. The critical behavior is instructive to that of the high-order cumulants of the net baryon number in the QCD phase transitions. Maybe it's difficult to distinguish the universality classes by the high-order cumulants in the heavy ion collisions.

1. Introduction

One of the primary goals of current ultra-relativistic heavy-ion collision experiments is to map the QCD phase diagram onto the T–μ_B plane [1]. The critical point is particularly interesting, because the divergent susceptibility and correlation length (ξ) are expected. But the expansion of ξ is limited as the finite evolution time and volume of relativistic heavy-ion collision system. So the more sensitive probes to locate the QCD critical point are needed. Recently, the high-order cumulants of the conserved charges are suggested, i.e., the net baryon number, net electric charge, and net strangeness [2, 3]. They are more sensitive to the correlation length and may change sign near the critical point based on the theory [4–7].

The net baryon number fluctuations have been studied by lattice QCD and QCD effective models [8–11]. However, owing to the difficulties of the lattice calculations and model estimations, the study of the high-order cumulants of the net baryon number need to be continued anyway [12, 13].

The QCD critical point falls into the same universality class with the 3D Ising model [14, 15]. In the chiral limit, the chiral phase transition for 2-flavor QCD is expected to belong to the 3D $O(4)$ universality class [15]. Because of the universal properties of the critical phenomena, the relevant cumulants can be studied in the simple spin systems. The results should be instructive to a finite system formed in relativistic heavy-ion collisions.

The paper is organized as follows, firstly, the cumulants of order parameter and energy from the $O(N)$ spin models are derived in Sec. 2. Then, their relations to the net baryon number fluctuations are discussed. In Sec. 3, the high-order cumulants from the 3D Ising model and $O(4)$ spin model are presented and discussed. Finally, the summary and conclusions are given in Sec. 4.

2. Cumulants in the $O(N)$ spin models

The $O(N)$ spin models are defined as,

$$\beta\mathcal{H} = -J\sum_{\langle i,j\rangle}\vec{S}_i\cdot\vec{S}_j - \vec{H}\cdot\sum_i\vec{S}_i, \tag{1}$$

where $\mathcal{H}$ is the Hamiltonian, J is an interaction energy between nearest-neighbor spins $\langle i,j\rangle$, and $\vec{H}$ is the external magnetic field. J and $\vec{H}$ are both reduced quantities which already contain a factor $\beta = 1/T$. $\vec{S}_i$ is a unit vector of N-components at site i of a d-dimensional hyper-cubic lattice. It is usually decomposed into the longitudinal (parallel to the magnetic field $\vec{H}$) and the transverse component $\vec{S}_i = S_i^{\parallel}\vec{e}_H + \vec{S}_i^{\perp}$, where $\vec{e}_H = \vec{H}/H$. For the 3D Ising and $O(4)$ spin models, $d = 3$, $N = 1$ and 4, respectively.

The partition function is as follows,

$$Z = \int\prod_i d^N S_i\delta(\vec{S}_i^2 - 1)\exp(-\beta E + HVS^{\parallel}), \tag{2}$$

where $E = -\sum_{\langle i,j\rangle}\vec{S}_i\cdot\vec{S}_j$ is the energy of a spin configuration, $S^{\parallel} = \frac{1}{V}\sum_i S_i^{\parallel}$ is the lattice average of the longitudinal spin components, $V = L^3$ and L is the number of lattice points of each direction.

As we know, the cumulants of the order parameter are related to the derivatives of the free energy density, $f(T,H) = -\frac{1}{V}\ln Z$, with respect to H. They can be got from the generating function [16],

$$\kappa_n^S = \frac{d^n}{dx^n}\ln\langle e^{xS^{\parallel}}\rangle\Big|_{x=0}. \tag{3}$$

So the first, second, third, fourth and sixth order cumulants of the order parameter are as follows,

$$\begin{aligned} \kappa_1^S &= \langle S^{\|} \rangle, \quad \kappa_2^S = \langle \delta S^{\|^2} \rangle, \quad \kappa_3^S = \langle \delta S^{\|^3} \rangle, \quad \kappa_4^S = \langle \delta S^{\|^4} \rangle - 3\langle \delta S^{\|^2} \rangle^2, \\ \kappa_6^S &= \langle \delta S^{\|^6} \rangle - 10\langle \delta S^{\|^3} \rangle^2 + 30\langle \delta S^{\|^2} \rangle^3 - 15\langle \delta S^{\|^4} \rangle\langle \delta S^{\|^2} \rangle, \end{aligned} \tag{4}$$

where $\delta S^{\|} = S^{\|} - \langle S^{\|} \rangle$, and κ_1^S is the magnetization (order parameter) of the system. At vanishing external magnetic field, due to the spatial rotation symmetry of the $O(N)$ groups, such defined order parameter is zero. In the case, an approximated order parameter definition is suggested as, $M = \langle |\frac{1}{V} \sum_i \vec{S}_i| \rangle$ [17].

The cumulants of the energy are related to the derivatives of the free energy density with respect to T. The forms of the formulas for the cumulants are the same as that in Eq. (4), where it just need to replace $S^{\|}$ by E.

In the vicinity of the critical point, the free energy density can be decomposed into two parts, the regular and singular parts. The critical related fluctuations are determined by the singular part. It has the scaling form $f_s(t,h) = l^{-d} f_s(l^{y_t} t, l^{y_h} h)$. Here $t = (T - T_c)/T_0$ and $h = H/H_0$ are reduced temperature and magnetic field, T_0 and H_0 are the normalized parameters. T_c is the critical temperature. y_t and y_h are universal critical exponents. In our simulation, we set $J = \beta$ and choose the approximate critical temperatures, T_c =4.51 [17] and 1.068 [18] for the 3D Ising and $O(4)$ spin models, respectively.

In order to map the result of the 3D Ising model to that of the QCD if the QCD critical point exists, the following linear ansatz is suggested [19–21],

$$t \approx T - T_{cp} + a(\mu - \mu_{cp}), \qquad h \approx \mu - \mu_{cp} + b(T - T_{cp}). \tag{5}$$

T_{cp} and μ_{cp} are the temperature and chemical potential at the QCD critical point, respectively. a and b are two undecided mixed parameters. The baryon-baryon correlation length diverges with the exponent y_t and exponent y_h when the critical point is approached along the t-direction and h-direction, respectively [9]. Because y_h (≈ 2.5) is larger than y_t (≈ 1.6) [22], the critical behavior of the net baryon number fluctuations is mainly controlled by the derivatives with respect to h, i.e., the fluctuations of the order parameter in the 3D Ising model.

The singular part of the free energy density for the chiral phase transition is suggested as [7]

$$\frac{f_s(T, \mu_q, h)}{T^4} = A h^{(1+1/\delta)} f_f(z), \qquad z = t/h^{1/\beta\delta}, \tag{6}$$

where β and δ are the universal critical exponents of the 3D $O(4)$ spin model. $f_f(z)$ is the scaling function. The reduced temperature t and external field h are expressed as follows,

$$t \equiv \frac{1}{t_0}(\frac{T - T_c}{T_c} + \kappa_\mu(\frac{\mu_q}{T})^2), \qquad h \equiv \frac{1}{h_0}\frac{m_q}{T_c}. \tag{7}$$

Here T_c is the critical temperature in the chiral limit. κ_μ is a parameter determined by QCD [23]. The net baryon number susceptibility is the derivative of free energy density with respect to the chemical potential μ_q. From Eqs. (6) and (7), it's clear that the form of the derivatives of the free energy density with respect to T and chemical potential $\hat{\mu}_q = \mu_q/T$ is similar. Particularly, The n-th order cumulant of the energy from the 3D $O(4)$ spin model is relevant to the $2n$-th (or n-th) order cumulant of the net baryon number at $\mu_q = 0$ (or $\mu_q \neq 0$) in the chiral phase transition.

3. Critical behavior of the high-order cumulants

The Monte Carlo simulations are performed by the Wolff algorithm with helical boundary conditions [24]. The typical size of an observable at a given magnetic field is determined by the saturation of size dependence, as shown in Ref. [25]. The system sizes in this work for each kind of cumulants at a given magnetic field and model are listed in Table 1.

Table 1. The typical system size for each kind of cumulant.

H	$\kappa_n^S(O(1))$	$\kappa_n^E(O(1))$	$\kappa_n^E(O(4))$
0	24	20	20
0.05	12	10	8
0.1	8	8	8

In order to compare the basic structure of the cumulants at different external fields, each cumulant is rescaled to unity by its maximum or minimum (except for the first order cumulant of the energy from the $O(4)$ spin model). For the Ising model, the cumulants of order parameter at $H = 0$ and $H \neq 0$ are quite different, so they are presented discretely.

The cumulants of energy from the 3D Ising model at $H = 0.1$, 0.05, and 0 are shown in the upper panel of Fig. 1. From each sub-figure, it's clear that the basic features of the cumulants, i.e., the patterns of the fluctuations, are not influenced by the magnitude of the external field. With the appearing or increasing of the external field, the whole critical region is amplified and shifted toward the higher temperature side. This is easy to understand that the cumulants are governed by universal functions that depend on the scaling variable $z = t/h^{1/\beta\delta}$.

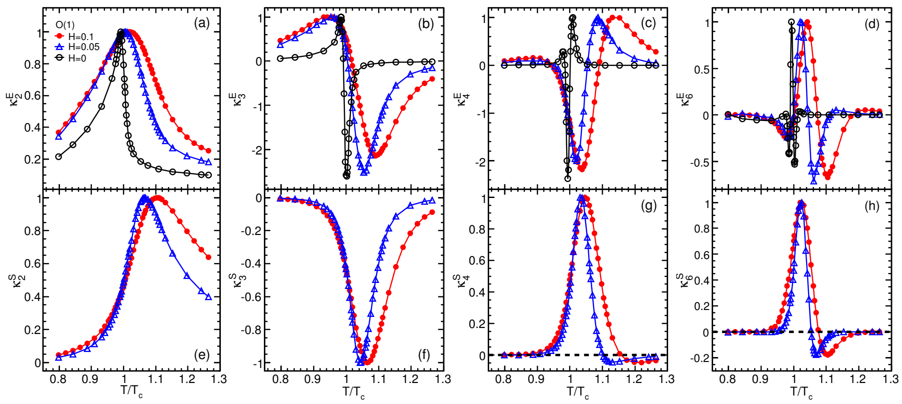

Fig. 1. (Color online) The 2nd, 3rd, 4th and 6th order cumulants of the energy (upper panel) and order parameter (lower panel) versus temperature for different H from the 3D Ising model.

In the vicinity of T_c, κ_2^E has a peak. κ_3^E oscillates and its sign changes from positive to negative when the temperature increases and passes the critical one. κ_4^E has two positive peaks locating at the two sides of T_c. The valley between the peaks is negative. In contrast to the κ_4^E, κ_6^E has two negative valleys and one positive peak in the vicinity of the critical temperature.

The cumulants of order parameter at $H = 0.1$ and 0.05 from the 3D Ising model are shown in the lower panel of Fig. 1. The influences of the external field are similar to those as discussed in the cumulants of the energy. In the vicinity of T_c, κ_2^S has a peak. κ_3^S has a negative valley and no sign change in the critical region. κ_4^S shows a obvious positive peak and a very small and negative valley when the temperature increases and passes the critical one. κ_6^S oscillates from positive to negative, and the negative valley is more obvious than that in κ_4^S.

Comparing the upper part and lower part of Fig. 1, it's clear that the generic structure of the same order cumulant of the energy is different from that of the order parameter, except for the second order one.

In Fig. 2, the high-order cumulants of the order parameter at $H = 0$ are shown. κ_2^S in Fig. 2(a) has a narrow and sharp peak near T_c. From Fig. 2(b) and (c), both κ_3^S and κ_4^S oscillate, but the former changes from negative to positive, while the latter changes from positive to negative with the increasing temperature. The generic structure of κ_6^S in Fig. 2(d) is similar to that of κ_4^E in Fig. 1(c). The behavior of the high-order cumulants at $H \neq 0$ and $H = 0$ is quite different. The sign change in the former case

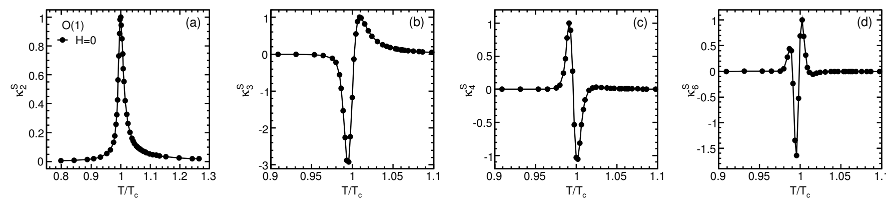

Fig. 2. The 2ed, 3rd, 4th and 6th order cumulants of the order parameter versus temperature at $H = 0$ from the 3D Ising model.

appears in the fourth order cumulant, while the third one in the later case.

The way of the sign change of κ_3^S at $H = 0$ as shown in Fig. 2(b) is consistent with the expectation from the effective model [5]. The qualitative features of κ_4^S as shown in Fig. 1(g) and Fig. 2(c) are consistent with that from the linear parametric model of the Ising universality class (see Fig. 1 in Ref. [6]). Especially at non-vanishing external field, the generic structure of κ_4^S in Fig. 1(g) is the same as Fig. 1(b) in Ref. [6]. When the critical point is approached from the higher temperature side, κ_4^S is negative. Based on the Ising universality class, the sixth order cumulant of the order parameter is also negative when approaching the critical point on the crossover side. And the negative valley is more obvious than that in the fourth order cumulant. Maybe the negative signal of the sixth order cumulant of the net baryon number is easier to be detected in the experiments.

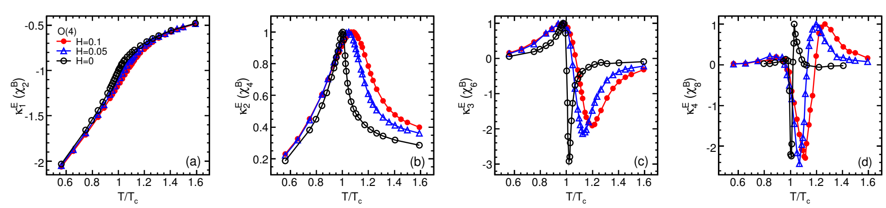

Fig. 3. (Color online) The 1st, 2ed, 3rd and 4th order cumulants of the energy versus temperature for different H from the 3D $O(4)$ spin model.

The cumulants of the energy from the 3D $O(4)$ spin model at $H = 0.1$, 0.05, and 0 are presented in Fig. 3. Again, the external field shows the similar influences as discussed above. κ_1^E increases with the temperature. κ_2^E, κ_3^E, and κ_4^E have a similar behavior with that from the Ising model.

As discussed in section 2, κ_1^E, κ_2^E, κ_3^E, and κ_4^E from the 3D $O(4)$ spin model are related to χ_2^B, χ_4^B, χ_6^B, and χ_8^B at $\mu_B = 0$ in the chiral phase transition. The positive peak of κ_2^E is consistent with χ_4^B from the lattice QCD calculations [8] and the estimations of QCD effective models [10, 11, 26]. The sign change of κ_3^E is also shown in χ_6^B from the Polyakov loop extended Quark-Meson (PQM) model [7, 11].

Based on the order parameter fluctuations from the Ising model at $H \neq 0$ and energy fluctuations from the $O(4)$ spin model, the generic structure of the fourth order cumulant of the net baryon number in the vicinity of the critical point and in the chiral phase transition at vanishing chemical potential is similar. Except for the small negative valley of the fourth order cumulant of the order parameter in the Ising model, they both have a obvious peak. The sixth order cumulant in the vicinity of the critical point is also similar to that in the chiral phase transition at vanishing chemical potential. It oscillates and has a sign change in the two cases. It's maybe difficult to distinguish the two universality classes by the high-order cumulants in the heavy ion collisions.

4. Summary

In this work, the behavior of the high-order cumulants of order parameter and energy in the 3D Ising model, and the cumulants of energy in the 3D $O(4)$ spin model at $H = 0.1$, 0.05, and 0 is presented, respectively. The external field does not influence the generic structure of the cumulants, except the cumulants of the order parameter from the 3D Ising model.

For the 3D Ising universality class, the generic structure of the high-order cumulants of energy are different from that of order parameter. But they all have the non-monotonic or sign change behavior. The fourth and sixth order cumulants of the order parameter at nonzero external field are both negative when approaching the critical point from the crossover side. But the negative signal is more obvious in the sixth order cumulant. Maybe it's easier to be detected in the experiment.

For the 3D $O(4)$ universality class, the behavior of the second to fourth order cumulants of energy is similar to that from the 3D Ising universality class. The net baryon number fluctuations based on the $O(4)$ spin model are qualitatively consistent with the calculations from the lattice QCD, and expectations from the QCD effective models. Our results also show that at vanishing chemical potential, the sixth order cumulant of the net baryon number is necessary in order to observe a sign change in the chiral phase transition.

Based on the order parameter fluctuations from the 3D Ising and energy fluctuations from the 3D $O(4)$ universality classes, maybe it's difficult to distinguish the different universality classes by the high-order cumulants in the heavy ion collisions.

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