Betti Numbers of Cut Ideals of Trees

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Abstract. Cut ideals, introduced by Sturmfels and Sullivant, are used in phylogenetics and algebraic statistics. We study the minimal free resolutions of cut ideals of tree graphs. By employing basic methods from topological combinatorics, we obtain upper bounds for the Betti numbers of this type of ideals. These take the form of simple formulas on the number of vertices, which arise from the enumeration of induced subgraphs of certain incomparability graphs associated to the edge sets of trees.

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1. Introduction

Let G be a simple graph. We denote the set of its vertices by $V(G)$ and the set of its edges by $E(G)$. By a *cut $A|B$* of G, we mean a partition of $V(G)$ into two subsets, $A, B \subseteq V(G)$ (so that $A \cap B = \emptyset$ and $A \cup B = V(G)$). Partitions are considered unordered, hence the number of cuts of a graph G is $2^{|V(G)|-1}$.

Note that a given cut $A|B$ also partitions the set of edges into two subsets: $S_{A|B} \subseteq E(G)$, consisting of those edges whose endpoints lie in different parts, and $T_{A|B} \subseteq E(G)$, consisting of those edges whose endpoints lie in the same part.

We may associate a toric ideal to a graph G as follows. Fix a field $\Bbbk$, and introduce the polynomial rings $R_G := \Bbbk[r_{A|B} : A|B \text{ a cut of } G]$ and $S_G := \Bbbk[s_e, t_e : e \in E(G)]$, in $2^{|V(G)|-1}$ and $2|E(G)|$ indeterminates, respectively. Define the ring homomorphism:

$$\phi_G : R_G \to S_G$$
$$r_{A|B} \mapsto \prod_{e \in S_{A|B}} s_e \cdot \prod_{e \in T_{A|B}} t_e.$$

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The *cut ideal* of a graph G is:

$$I_G := \ker \phi_G \subset R_G.$$

Cut ideals were introduced by Sturmfels and Sullivant in [9]. They showed how to obtain generators for the cut ideal of a graph G in terms of the generators of two cut ideals I_{G_1} and I_{G_2} in the case where G is a zero-, one- or two-sum of two graphs G_1 and G_2. They also obtained necessary and sufficient conditions for the toric variety $\mathbf{V}(I_G)$ to be smooth, among other results. Later, Nagel and Petrović [6] proved that the cut ideals of ring graphs admit a quadratic Gröbner basis and established algebraic properties which derive from this fact, such as Cohen-Macaulayness and Koszulness of the coordinate ring R_G/I_G. In [3], Engström proved a conjecture presented in [9], namely that the cut ideals of K_4-minor free graphs are generated by quadrics.

We shall study minimal free resolutions of the cut ideals of the class of *tree graphs*. A tree graph is a connected graph without cycles. From [6], we know that their cut ideals are arithmetically Gorenstein, and from [3] that they are generated by quadrics. We denote a tree on n vertices by T_n. By applying some topological ideas introduced by Engström and Dochtermann in [2], we were able to obtain estimates for some of the Betti numbers of the cut ideals of T_n:

Theorem 1.1. *For the Betti numbers of $I_{T_{n+1}}$, we have the following bounds, which hold independently of the underlying field:*

$$\beta_{0,2}(I_{T_{n+1}}) \leq \beta_{0,2}(\text{in } I_{T_{n+1}}) = \frac{1}{2}\left(4^n - 2\cdot3^n + 2^n\right)$$

$$\beta_{1,3}(I_{T_{n+1}}) \leq \beta_{1,3}(\text{in } I_{T_{n+1}}) = \frac{1}{3}\left(8^n - 3\cdot6^n + 3\cdot4^n - 2^n\right)$$

$$\beta_{2,4}(I_{T_{n+1}}) \leq \beta_{2,4}(\text{in } I_{T_{n+1}}) = \frac{1}{8}\left(16^n - 4\cdot12^n + 6\cdot8^n + 2\cdot7^n - 4\cdot6^n + 4\cdot5^n\right.$$
$$\left. -9\cdot4^n + 2\cdot3^n + 2\cdot2^n\right)$$

$$\beta_{1,4}(I_{T_{n+1}}) \leq \beta_{1,4}(\text{in } I_{T_{n+1}}) = \frac{1}{4}\left(7^n - 4\cdot6^n + 6\cdot5^n - 4\cdot4^n + 3^n\right)$$

$$\beta_{3,5}(I_{T_{n+1}}) \leq \beta_{3,5}(\text{in } I_{T_{n+1}}) = \frac{1}{60}\left(2\cdot32^n - 10\cdot24^n + 30\cdot20^n - 120\cdot18^n + 30\cdot17^n\right.$$
$$-40\cdot16^n + 180\cdot15^n + 375\cdot14^n - 420\cdot13^n - 180\cdot12^n$$
$$+200\cdot11^n - 280\cdot10^n - 220\cdot9^n + 985\cdot8^n - 720\cdot7^n$$
$$\left.+655\cdot6^n - 710\cdot5^n + 35\cdot4^n + 340\cdot3^n - 132\cdot2^n\right)$$

$$\beta_{2,5}(I_{T_{n+1}}) \leq \beta_{2,5}(\text{in } I_{T_{n+1}}) = \frac{1}{12}\left(3\cdot14^n - 12\cdot12^n - 2\cdot11^n + 22\cdot10^n - 2\cdot9^n\right.$$
$$\left.-9\cdot8^n - 6\cdot7^n + 9\cdot6^n - 10\cdot5^n + 11\cdot4^n - 4\cdot3^n\right)$$

We emphasize that, since the main algebraic results used to obtain these bounds, that is Hochster's formula (cf. Theorem 1.2 below) and the upper semi-continuity of Betti numbers (cf. Lemma 2.2), apply for an arbitrary field, our arguments adapt almost directly

to obtaining bounds for the Betti numbers of the cut ideals of trees over any field (see the last paragraph in the proof of Theorem 1.1).

1.1. Definitions

We recall some definitions that are necessary for presenting our proof of Theorem 1.1. A *simplicial complex* Δ is a collection of subsets of a finite base set $[n] := \{1, 2, \ldots, n\}$ which is closed under taking subsets. The *independence complex* $\mathsf{Ind}(G)$ of a graph G is the simplicial complex on $V(G)$ whose faces are the subsets of $V(G)$ which are not adjacent in G. The *Stanley-Reisner ideal* of a simplicial complex Δ on n vertices is the square-free monomial ideal:

$$I_\Delta := \langle x_{i_1} x_{i_2} \ldots x_{i_r} : (i_1, i_2, \ldots, i_r) \notin \Delta \rangle \subseteq \Bbbk[x_1, \ldots, x_n].$$

One of the cornerstones of combinatorial commutative algebra is *Hochster's formula*, relating the Betti numbers of the minimal free resolution of a Stanley-Reisner ideal I_Δ with the homology of the subcomplexes of Δ:

Theorem 1.2 (Hochster's formula, [5])**.** *For $i > 0$, the Betti numbers $\beta_{i,j}$ of the Stanley-Reisner ideal of a simplicial complex Δ are given by:*

$$\beta_{i,j}(I_\Delta) = \sum_{\substack{F \subseteq V(\Delta) \\ |F| = j}} \dim_\Bbbk \tilde{H}_{j-i-1}(\Delta[F], \Bbbk), \qquad (1)$$

where $\Delta[F]$ refers to the subcomplex of Δ induced by the vertices in F.

The *edge ideal* of a graph G on n vertices is the monomial ideal defined as $\langle x_i x_j : (ij) \in E(G) \rangle \subseteq \Bbbk[x_1, \ldots, x_n]$.

2. Proof of the Main Result

A couple of preliminary computations using `Macaulay2`[4] provide us with the first two total Betti numbers of the cut ideals, over $\mathbb{Q}$, for some path graphs P_n. They are listed in Table 2.1[†].

We observe that the number of generators and first syzygies increases quickly when considering larger paths. Accordingly, direct use of the usual functions in `Macaulay2` (for example, `gens`, `syz`, `res`) becomes unfeasible when doing the computations. This motivates the general strategy of trying to relate combinatorial properties of the graphs to algebraic properties of their cut ideals, which we shall follow now.

Our approach derives from the results of Engström and Dochtermann in [2] and consists of two steps:

[†]The blank in the last entry means the computation had not concluded after roughly one hour. For the next heaviest tasks, the computation of the syzygies of I_{P_7} and the generators of I_{P_8}, the CPU time was 55.7 minutes and 8.8 seconds, respectively. We used a 8 AMD Opteron Dual-Core 2.6 GHz computing server with 64 GB RAM running Ubuntu 12.04. In each case, the generators were obtained with the `4ti2`[1] interface for `Macaulay2`.

	β_0	β_1
I_{P_3}	1	0
I_{P_4}	9	16
I_{P_5}	55	320
I_{P_6}	285	4160
I_{P_7}	1351	44800
I_{P_8}	6069	-

Table 2.1: First two total Betti numbers of cut ideals of path graphs.

- Construct a Gröbner basis for I_{T_n}, and characterize the initial ideal from which it arises combinatorially.

- Regard the initial ideal gotten as the Stanley-Reisner ideal of a certain simplicial complex, and use *Hochster's formula* to get an estimate for the Betti numbers of I_{T_n}.

To have a picture of the initial ideals associated to I_{T_n}, we present the complete Betti diagrams for their minimal free resolutions in Table 2.2 for $n = 4$ and $n = 5$. Here we use the standard monomial ordering from `Macaulay2`, namely graded reverse lexicographic.

```
               total: 1 9 16 9 1
                  0: 1 .  . . .
   n = 4          1: . 9 16 9 .
                  2: . .  . . 1
```

```
        total: 1 55 326 951 1744 2273 2273 1744 951 326 55  1
           0: 1  .   .   .    .    .    .    .   .   .   .  .
n = 5      1: . 55 320 897 1462 1437  836  282  54   6   .  .
           2: .  .   6  54  282  836 1437 1462 897 320 55  .
           3: .  .   .   .    .    .    .    .   .   .   .  1
```

Table 2.2: Betti diagrams for the initial ideals of the cut ideals of paths on four and five vertices.

We start with an observation. Note that the identification $A|B \mapsto S_{A|B}$ defines an injective mapping from the set of cuts of G into $\mathbf{2}^{E(G)}$. If G is a tree, this map is a surjection onto $\mathbf{2}^{E(G)}$, and hence every subset of $E(G)$ corresponds bijectively to a cut of G. This allows us to think of the indeterminates in R_G equivalently as being labelled by the subsets of $E(G)$ (see Figure 2.1a).

We wish to introduce a *normal form* for the monomials in R_{T_n}. To describe it, picture a monomial m as stacked subsets of edges; then the normal form of m is obtained by *sending edges to the bottom* as illustrated in Figure 2.1b. As it turns out, the normal form of a monomial can be reached in steps by successively sending edges to the bottom for pairs of cuts in the monomial. The translation into algebraic terms goes as follows.
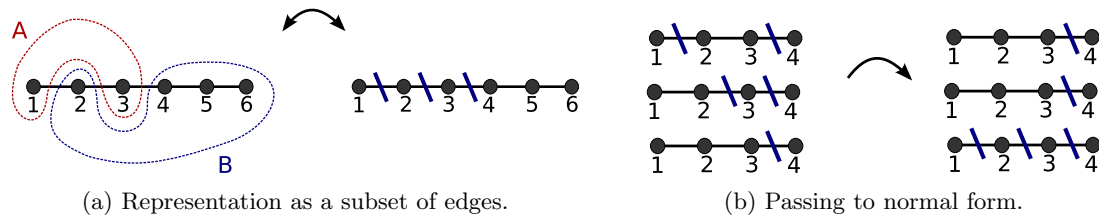
(a) Representation as a subset of edges.   (b) Passing to normal form.

Figure 2.1: Equivalence of cuts and subsets of edges.

Proposition 2.1. *Let I_{T_n} be the cut ideal associated to a tree T_n. Then there is a term order in R_{T_n} with respect to which the set*

$$\mathcal{B} = \{r_X \cdot r_Y - r_{X \cup Y} \cdot r_{X \cap Y} : X, Y \subset E(T_n) \text{ are incomparable}\} \subset I_{T_n}$$

is a Gröbner basis for I_{T_n} (here we designate a cut of T_n by the subset of $E(T_n)$ it cuts).

Proof. (Adapted from the proof of Theorem 9.1 in [8].) Assign a weight to the indeterminate r_X as the number of elements $X' \subset E(T_n)$ incomparable with X. Let $\prec$ be any term order refining the partial order given by the weights to a total order. We claim that $\mathcal{B}$ is a Gröbner basis for I_{T_n} with respect to $\prec$. Assume to the contrary, that there is at least one binomial $b = m - m' \in I_{T_n}$ such that m is not divisible by any $r_X \cdot r_Y$ with $X, Y \subset E(T_n)$ incomparable. This implies that all indeterminates appearing in m are mutually comparable.

By taking b to be the minimal binomial providing a counterexample, we may suppose that m and m' have no common factors, and further, that the (labels of the) indeterminates in m' are also mutually comparable (otherwise we may reduce m' modulo $\mathcal{B}$ to have it that way). Then m and m' have disjoint sets of indeterminates. But since b belongs to I_{T_n}, m and m' must cut the same edges the same number of times (and equally for edges kept together). This can only happen if $m = m'$, because only then can the indeterminates inside m and m' be mutually comparable, thus leading us to a contradiction. Hence, m must be divisible by some $r_X \cdot r_Y$ with $X, Y \subset E(T_n)$ incomparable.

We will use the next lemma towards the end of the proof of Theorem 1.1. The reader can refer to [5] for further details.

Lemma 2.2 (Upper semicontinuity, Theorem 8.29 in [5])**.** *Fix a graded ideal I in a polynomial ring $\Bbbk[x_1, \ldots, x_n]$. If $\mathrm{in}(I)$ is the initial ideal of I with respect to some term order, then:*

$$\beta_{i,j}(I) \leq \beta_{i,j}(\mathrm{in}(I)) \text{ for all } i, j \in \mathbb{N}$$

Proof. (Theorem 1.1) Let $\mathrm{in}\, I_{T_n}$ be the initial ideal giving rise to the Gröbner basis of I_{T_n} presented in Proposition 2.1. This monomial ideal can be regarded as the *edge ideal* of a graph $\Gamma_{E(T_n)}$ with the elements of $\mathbf{2}^{E(T_n)}$ as the vertex set and the pairs of the
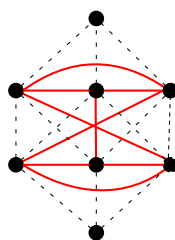
Figure 2.2: $\Gamma_{E(T_n)}$ for $n = 4$.

incomparable subsets of $E(T_n)$ as the edge set. We call $\Gamma_{E(T_n)}$ the *incomparability graph of* $\mathbf{2}^{E(T_n)}$; clearly, it only depends on $|E(T_n)| = n - 1$.

Now, extracting the insight from [2], we consider in I_{T_n} as the Stanley-Reisner ideal of the simplicial complex $\mathsf{Ind}(\Gamma_{E(T_n)})$. This means that we can use Hochster's formula to calculate the Betti numbers of in I_{T_n} with knowledge of the counts of induced subgraphs of $\Gamma_{E(T_n)}$ and the (dimension of the) reduced homology of their independence subcomplexes:

$$\beta_{i,j}\left(\text{in } I_{T_n}\right) = \beta_{i,j}\left(I_{\mathsf{Ind}(\Gamma_{E(T_n)})}\right) = \sum_{\substack{F \subseteq V(\Gamma_{E(T_n)}) \\ |F| = j}} \dim_{\Bbbk}\left(\tilde{H}_{j-i-2}(\mathsf{Ind}(\Gamma_{E(T_n)}[F]))\right). \quad (2)$$

The enumeration of the induced subgraphs of $\Gamma_{E(T_n)}$ is a straightforward combinatorial calculation, which can be performed using inclusion-exclusion. We implemented this procedure in a `Python` script [7] to get formulas for the number of induced subgraphs, and illustrate it with the enumeration of $\diagdown\!\diagdown$.

We write the elements of $\Gamma_{E(T_n)}$ as tuples in $\{0, 1\}^{n-1}$. Then, we can compare $X, Y, Z \in \Gamma_{E(T_n)}$ by indicating the number of entries i for which (X_i, Y_i, Z_i) attains every possible value. Let these numbers be $a, b, \ldots, h$, as in Figure 2.3.

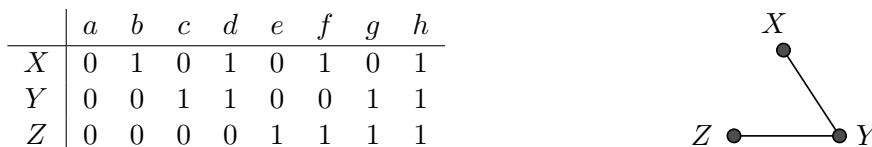|	a	b	c	d	e	f	g	h
X	0	1	0	1	0	1	0	1
Y	0	0	1	1	0	0	1	1
Z	0	0	0	0	1	1	1	1

Figure 2.3: Comparison of $X, Y, Z \in \Gamma_{E(T_n)}$.

The condition for the edge XY to be present, for instance, is that both $b + f$ and $c + g$ be larger than zero. Hence, the number of labelled occurences of $\diagdown\!\diagdown$ in $\Gamma_{E(T_n)}$ is given by the sum:

$$\sum_{\substack{a+b+\ldots+h=n-1, \\ b+f>0,\ c+g>0, \\ c+d>0,\ e+f>0, \\ b+d=0 \text{ xor } e+g=0}} \binom{n-1}{a, b, \ldots, h} = \sum_{\substack{a+c+e+f+g+h=n-1, \\ f>0,\ c+g>0, \\ c>0,\ e+f>0, \\ b+d=0}} \binom{n-1}{a, c, e, f, g, h} + \ldots$$

$$\sum_{\substack{a+b+c+d+f+h=n-1, \\ b+f>0,\; c>0, \\ c+d>0,\; f>0, \\ e+g=0}} \binom{n-1}{a,b,c,d,f,h} - 2 \sum_{\substack{a+c+f+h=n-1, \\ f>0,\; c>0, \\ b+d=0, e+g=0}} \binom{n-1}{a,c,f,h}.$$

The terms on the right hand side are decomposed according to inclusion-exclusion:

$$\sum_{\substack{a+c+f+h=n-1 \\ f>0}} = \sum_{a+c+f+h=n-1} - \sum_{\substack{a+c+h=n-1 \\ f=0}},$$

and by taking symmetry into account, we obtain:

$$\#\lambda(\Gamma_{E(T_n)}) = 6^{n-1} - 2\cdot 5^{n-1} + 2\cdot 3^{n-1} - 2^{n-1}.$$

Finally, Table 2.3 contains the (dimensions of the reduced) homologies of the independence complexes of the induced subgraphs making contributions. Note that, as long as the independence complexes of the graphs involved in equation (2) have torsion-free homology groups, we do not have to be concerned about the characteristics of the underlying field $\Bbbk$. Putting this together with the counts for the induced subgraphs, we obtain the formulas in Theorem 1.1. The fact that these expressions bound the Betti numbers of I_{T_n} from above is a consequence of the well-known upper semicontinuity for the Betti numbers of a minimal free resolution, stated in Lemma 2.2. Thus, we conclude the proof.

Graph	dim $\tilde{H}_0$	dim $\tilde{H}_1$
(graph)	1	0
(graph)	2	0

Graph	dim $\tilde{H}_0$	dim $\tilde{H}_1$
(graph)	3	0
(graph)	2	0
(graph)	1	0
(graph)	1	0
(graph)	1	0
(graph)	0	1

Graph	dim $\tilde{H}_0$	dim $\tilde{H}_1$	Graph	dim $\tilde{H}_0$	dim $\tilde{H}_1$
(graph)	4	0	(graph)	3	0
(graph)	2	0	(graph)	2	0
(graph)	1	0	(graph)	0	1
(graph)	1	0	(graph)	1	0
(graph)	2	0	(graph)	1	0
(graph)	1	1	(graph)	0	1
(graph)	1	0	(graph)	0	1
(graph)	0	2	(graph)	1	0
(graph)	1	0	(graph)	0	1

Table 2.3: Contributions to the reduced homology of the independence complexes of graphs over $\mathbb{Z}$. Notice that there is no torsion in any of these homology groups.

Below, in Table 2.4, are the estimates for the Betti numbers of I_{T_n} for n up to nine.

n	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\beta_{1,4}$	$\beta_{2,5}$	$\beta_{3,5}$
3	1	0	0	0	0	0
4	9	16	9	0	0	0
5	55	320	897	6	54	1450
6	285	4160	32025	150	3380	156824
7	1351	44800	810255	2280	115950	9798758
8	6069	435356	17298519	27300	2984380	474814396
9	26335	3978240	335187657	283626	64924734	19911592842

Table 2.4: The Betti numbers of the initial ideals. Obtained by using subgraph counts and the homology of independence complexes.

3. Further remarks

A few interesting questions were left unaddressed in this note.

1. Does the combinatorial description we used for the initial ideal of I_{T_n} provide information about the cellular complexes supporting the minimal cellular resolution of in I_{T_n}? Our initial computations of the minimal free resolutions of in I_{T_n} with `Macaulay2` showed that the minimal free resolution of the cut ideal of a tree on four vertices is supported by the polytopal complex in Figure 3.1.
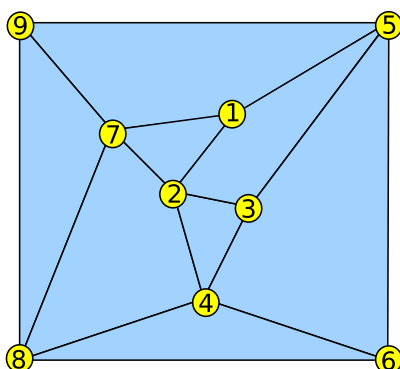

Figure 3.1: A Schlegel diagram of a polytopal complex supporting the minimal free resolution of the initial monomial ideal of P_4. The monomials corresponding to the vertices are, respectively, $r_{1|234}r_{12|34}$, $r_{1|234}r_{123|4}$, $r_{12|34}r_{123|4}$, $r_{2|134}r_{123|4}$, $r_{12|34}r_{14|23}$, $r_{2|134}r_{14|23}$, $r_{1|234}r_{124|3}$, $r_{2|134}r_{124|3}$, $r_{14|23}r_{124|3}$.

2. Do our methods allow us to establish the polytopality of the supporting cell complexes for the cut ideals of trees of any size? Do they allow to establish the symmetry of the f-vector of such cellular complexes (which would reflect the arithmetic-Gorenstein nature of the cut ideals of trees, which was established in [6])?

3. As a last comment, we mention that a computation of the minimal free resolution of the cut ideal of a tree on five vertices yielded the non-unimodal Betti vector $(1, 55, 320, 891, 1436, 1375, 1375, 1436, 891, 320, 55, 1)$. As seen in Table 2.2, this property washed away when looking at the minimal free resolution of an initial ideal. Is it possible to recover a non-unimodal Betti vector from the initial ideals of the cut ideals of trees with respect to some term order? This would amount to a cell complex with a non-unimodal f-vector supporting a minimal cellular resolution of the initial ideal. That would be of independent interest in the case that the cell complexes from the previous paragraph are polytopal.

Acknowledgements

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