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The degeneration of the Grassmannian into a toric variety and the calculation of the eigenspaces of a torus action

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Abstract. Using the method of degenerating a Grassmannian into a toric variety, we calculate formulas for the dimensions of the eigenspaces of the action of an n-dimensional torus on a Grassmannian of planes in an n-dimensional space.

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1. Introduction

An n-dimensional torus acts on a Grassmannian of planes in an n-dimensional linear space in a natural way. The main purpose of this paper is to show a method of calculating the dimensions of the eigenspaces of the induced action on the coordinate ring of the Grassmannian (Theorem 3). The result is presented as a generating function (the Poincare-Hilbert series).

The calculations in this paper are based on the degeneration of the Grassmannian into the toric variety described by a 3-valent tree, which makes it possible to reinterpret algebraic properties of the Grassmannian in the language of combinatorics. The formulas for the dimensions of the eigenspaces are then derived using characteristics of 3-valent trees. To simplify the calculations, I focus on toric varieties described by a caterpillar tree (see Figure 2).

It's worth mentioning that the computed dimensions are equal to the dimensions of the spaces of global sections of invertible sheaves on $n-3$ dimensional projective spaces with $n-1$ blown-up points (see the introduction to [10]). It follows from the fact that the coordinate ring of a Grassmannian may be interpreted as the ring of total coordinates of this variety. I elaborate more on this in Remark 2.

Furthermore, the construction of the toric varieties described by a 3-valent tree comes from analysing Markov processes on trees and is connected with phylogenetics, the branch

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of science which tries to understand the evolution of species. It is important in applications to calculate the invariants of varieties representing statistical models of evolution. General information about mathematical aspects of phylogenetics may be found in [11] and an introduction to algebraic methods in [5].

Closed formulas for the sought-for dimensions have been established by Hering and Howard (see [8]) around the time the first version of this paper was written. For toric degenerations, I refer to [12] and [6, Theorem 7.35]. Readers interested in applications of the degeneration of the Grassmaniann into toric varieties are encouraged to also consult [9]. One of the sources of inspiration for my paper was [2].

The paper is organised in the following way.

In the second section, which consists of well-known facts, I introduce the aforementioned action of a torus and present the standard method of degenerating the Grassmannian of planes in an n-dimensional linear space into the toric variety described by a 3-valent tree with n leaves. The dimensions I am looking for are invariant under this degeneration. Further, I discuss characteristics of those toric varieties.

In the third section, given that the sought-for dimensions are determined by the properties of the semigroup of the toric variety described by a 3-valent tree, I single out a special presentation of elements of this semigroup. Subsequently, I use this presentation together with inclusion-exclusion principle to find a formula for the Poincare-Hilbert series (Theorem 3). Additionally, I show a recursive formula for the Poincare-Hilbert series using independent combinatorial arguments (Theorem 4). I also present my recursive formula for the numerator of the Poincare-Hilbert series, but the technical proof is not attached in the paper.

2. Preliminaries

In this paper we assume that the reader knows the notation and basic facts from algebraic geometry, toric geometry and graph theory. The notions may be found in [7], [3] (first chapter) and [11] respectively. This section consists of material which is well known, so we omit the majority of proofs.

Trees (acyclic connected graphs) whose vertices have degree three will be called 3-*valent trees*. We assume that every tree has a fixed embedding in the plane such that its edges do not intersect. Additionally, we assume that leaves lie on a circle. This will be required to number the leaves.

Let A be a ring with $\mathbb{Z}^n$-grading where A_0 is a field and let M be an A-module with $\mathbb{Z}^n$-grading. We define its *Poincare-Hilbert series* to be:

$$W(M) = \sum_{\lambda \in \mathbb{Z}^n} \dim(M_\lambda) z^\lambda \in \mathbb{Z}[\![z_1, z_2, \ldots, z_n]\!],$$

where M_λ is a linear space of elements of M of grading λ. For an element $\lambda = (\lambda_1, \ldots, \lambda_n)$ of the lattice $\mathbb{Z}^n$ and for the ring of polynomials of n variables $\mathbb{Z}[z_1, \ldots, z_n]$, we let z^λ denote $z_1^{\lambda_1} \ldots z_n^{\lambda_n}$.

2.1. Toric geometry

The following notions, definitions and propositions can be found in [3]. An affine variety T which is isomorphic to $(\mathbb{C}^*)^n$, will be called a *torus*. A homomorphism $\chi\colon T \to \mathbb{C}^*$ of algebraic groups, will be called a *character of the torus* T. The group of characters $\mathrm{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*)$ of the torus T will be denoted by M_T and called the *character lattice* of T. It is isomorphic to $\mathbb{Z}^n$, where n is the dimension of the torus.

An affine irreducible variety V will be called an *affine toric variety* if some Zariski open subset of it is isomorphic to a torus and the standard action of this torus on itself extends to an algebraic action on the variety V.

Let T be a torus and let M_T be its character lattice. A finite subset $\mathcal{A} \subset M_T$ defines the toric variety $\mathrm{Spec}(\mathbb{C}[\mathbb{Z}_{\geq 0}\mathcal{A}])$ (see [3, Definition 1.1.7 and Proposition 1.1.14]). Its torus has character lattice $\mathbb{Z}\mathcal{A}$ (see [3, Proposition 1.1.8]). We call $\mathbb{Z}_{\geq 0}\mathcal{A}$ the *semigroup* of the affine toric variety associated to $\mathcal{A}$.

2.2. The Grassmannian of planes

In this paper we analyse the variety $\mathcal{G}$ in the space $\mathbb{P}_{\mathbb{C}}^{n(n-1)/2}$ with coordinates $x_{i,j}$ for $1 \leq i < j \leq n$, given by the equations:

$$x_{i,j}x_{k,l} - x_{i,k}x_{j,l} + x_{i,l}x_{j,k} = 0 \text{ for } 1 \leq i < j < k < l \leq n.$$

The algebra of this projective variety will be denoted $\mathbb{C}[\mathcal{G}] := \mathbb{C}[x_{i,j}]/I$, where I is the ideal generated by the preceding polynomials. The variety $\mathcal{G}$ is a Grassmannian of planes in an n-dimensional linear space. The equations come from the Plücker embedding.

We introduce the action of the torus $(\mathbb{C}^*)^n$ on the variety $\mathcal{G}$. Let $\sigma_t = (t_1, t_2, \ldots, t_n) \in (\mathbb{C}^*)^n$ act on coordinates in the following way:

$$\sigma_t(x_{i,j}) = t_i t_j x_{i,j}.$$

The action of the torus defines a natural $\mathbb{Z}^n$-grading on $\mathbb{C}[\mathcal{G}]$. The element $f \in \mathbb{C}[\mathcal{G}]$ is in a grading $\lambda \in \mathbb{Z}^n$ if for any $\sigma_t \in (\mathbb{C}^*)^n$ we have

$$\sigma_t(f) = t^\lambda f,$$

where $t^\lambda := t_1^{\lambda_1} \ldots t_n^{\lambda_n}$. This action is well defined on the quotient ring $\mathbb{C}[\mathcal{G}]$, because $\sigma_t(x_{i,j}x_{k,l}) = t_i t_j t_k t_l x_{i,j}x_{k,l}$, and so generators of the ideal consist of polynomials which belong to the same grading.

Our main goal is to calculate the Poincare-Hilbert series of the algebra $\mathbb{C}[\mathcal{G}]$ for the grading described above. The following well-known proposition holds:

Proposition 1. *Let A be a finitely generated A_0 algebra, where A_0 is a field, generated by elements $a_1, \ldots, a_k$ which belong to gradings $\lambda_1, \ldots, \lambda_k \in \mathbb{Z}^n$ respectively. Let M be finitely generated over A. Then the Poincare-Hilbert series can be written in the following form:*

$$\frac{F(z_1, \ldots, z_n)}{\prod_{i=1}^k (1 - z^{\lambda_i})}, \text{ where } F \in \mathbb{Z}[z_1, \ldots, z_n].$$

The proof of this proposition repeats the argument from [1, Theorem 11.1] applied to a multidimensional grading.

The coordinate $x_{i,j}$ of the variety $\mathcal{G}$ belongs to the grading $t_i t_j$. Therefore, the Poincare-Hilbert series of the Grassmannian can be written in the following form:

$$\frac{F(z_1,\ldots,z_n)}{\prod_{1\le i<j\le n}(1-z_iz_j)}, \text{ where } F\in\mathbb{Z}[z_1,\ldots,z_n].$$

2.3. The variety described by a tree

For this subsection, see also [9] and [12]. Let us recall that we assume that each tree has a fixed embedding in the plane in which edges do not intersect and leaves lie on a circle.

Let $\mathcal{T}$ be some 3-valent tree with n leaves for $n>2$. It has $2n-2$ vertices, $2n-3$ edges and contains a vertex which is a neighbour of two leaves. We number leaves (anticlockwise around the tree) and edges with consecutive natural numbers starting from one (see for instance Figure 3). Between each pair of leaves there exists a unique shortest path. For two leaves $i<j$ of the tree $\mathcal{T}$, we define a vector $w_{i,j}\in\mathbb{Z}^{2n-3}$ such that it has one at the kth coordinate, if the kth edge lies on the shortest path between ith and jth leaves, and zero otherwise. To ease the notation, we also denote by $w_{i,j}$ the path between ith and jth leaves.

Define $\mathcal{A}_{\mathcal{T}} := \{w_{i,j} \mid 1\le i<j\le n\}\subseteq\mathbb{Z}^{2n-3}$ and let $X_{\text{aff}}(\mathcal{T})$ be the toric variety associated to this subset of the character lattice $\mathbb{Z}^{2n-3}$ of the torus $(\mathbb{C}^*)^{2n-3}$ (see Subsection 2.1). This variety is a cone, and so it induces a projective variety $X(\mathcal{T})$ in $\mathbb{P}_{\mathbb{C}}^{n(n-1)/2-1}$, which we will call the *variety described by the tree* $\mathcal{T}$. Further, the affine variety $X_{\text{aff}}(\mathcal{T})$ is normal. We denote by $x_{i,j}$ the coordinate of $X_{\text{aff}}(\mathcal{T})$ corresponding to $w_{i,j}$.

The character lattice of $X_{\text{aff}}(\mathcal{T})$ is equal to $\mathbb{Z}\mathcal{A}_{\mathcal{T}}$ and the semigroup $S(\mathcal{T})$ of $X_{\text{aff}}(\mathcal{T})$ is equal to $\mathbb{Z}_{\ge 0}\mathcal{A}_{\mathcal{T}}$. In informal words: we can perceive elements of the character lattice $\mathbb{Z}\mathcal{A}_{\mathcal{T}}$ of $X_{\text{aff}}(\mathcal{T})$ as such assignments of integers to edges which come from adding and subtracting paths (as vectors) and we can perceive elements of the semigroup $S(\mathcal{T})$ as such assignments of non-negative integers to edges, which come from just adding paths. The operation in this semigroup is adding numbers on corresponding edges.

Remark 1. *One can check that elements of the semigroup $S(\mathcal{T})$ are exactly the assignments of nonnegative integers such that for every node of $\mathcal{T}$ the numbers on its adjacent edges satisfy triangle inequality and their sum is even.*

We introduce the following natural metric on vertices of a tree. The distance between two vertices is the number of edges in the shortest path which connects those vertices. We denote the distance between the vertices i and j by $d(i,j)$.

We prove the following lemma, because we will use similar ideas later in the paper.

Lemma 1. *Let $1\le i<j<k<l\le n$. Then either $w_{i,j}$ intersects $w_{k,l}$, or $w_{i,l}$ intersects $w_{j,k}$. The path $w_{i,k}$ always intersects $w_{j,l}$.*

Proof. Let v be the vertex in the intersection of $w_{i,j}, w_{i,k}$ and $w_{i,l}$, which has the greatest distance from i. Clearly, $v \neq i, j, k$. The vertex v splits the tree into three subtrees: $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$, numbered anticlockwise in such a way that i belongs to $\mathcal{T}_1$. Due to the order of leaves, j must belong to $\mathcal{T}_2$ and l to $\mathcal{T}_3$. We consider two cases.

Case 1. Assume the vertex k lies in $\mathcal{T}_2$.

Then the path $w_{i,l}$, which is disjoint with $\mathcal{T}_2$, does not intersect $w_{j,k}$, which belongs to $\mathcal{T}_2$. On the other hand, $w_{i,j}$ intersects $w_{k,l}$, since the edge from v to $\mathcal{T}_2$ lies in both.

Case 2. Assume the vertex k lies in $\mathcal{T}_3$.

Then the path $w_{i,j}$, which is disjoint with $\mathcal{T}_3$, does not intersect $w_{k,l}$, which belongs to $\mathcal{T}_3$. On the other hand, $w_{i,l}$ intersects $w_{j,k}$, since the edge from v to $\mathcal{T}_3$ lies in the intersection.

The paths $w_{i,k}$ and $w_{j,l}$ contain the vertex v, so they intersect.

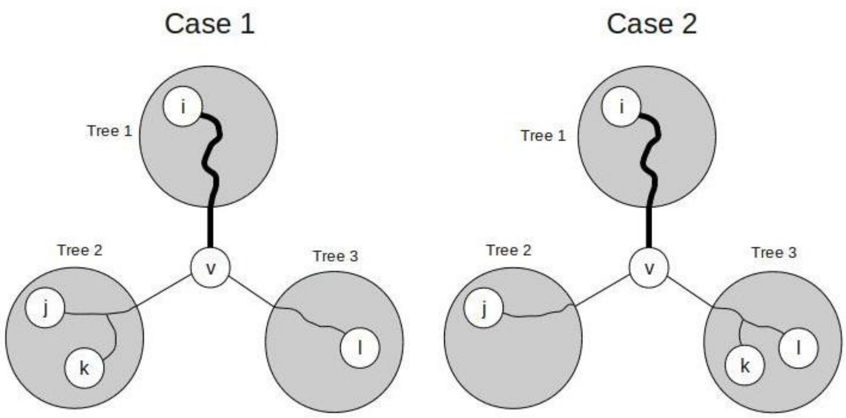

Figure 1: Two possibilities of intersection

□

The lemma implies the following corollary:

Corollary 1. *The following inequalities hold:*

1. $d(i,l) + d(j,k) < d(i,j) + d(k,l) = d(i,k) + d(j,l)$, *if* $w_{i,j}$ *intersects* $w_{k,l}$,
2. $d(i,j) + d(k,l) < d(i,l) + d(j,k) = d(i,k) + d(j,l)$, *if* $w_{i,l}$ *intersects* $w_{j,k}$.

Proposition 2 ([9, Proposition 3.1])**.** *For every four numbers* $1 \leq i < j < k < l \leq n$ *one of the following polynomials lie in the ideal of* $X_{\mathrm{aff}}(\mathcal{T})$*:*

1. $W_1(i,j,k,l) = x_{i,j}x_{k,l} - x_{i,k}x_{j,l}$, *if* $w_{i,j}$ *intersects* $w_{k,l}$,
2. $W_2(i,j,k,l) = x_{i,l}x_{j,k} - x_{i,k}x_{j,l}$, *if* $w_{i,l}$ *intersects* $w_{j,k}$.

The polynomials given above generate the ideal of $X_{\mathrm{aff}}(\mathcal{T})$.

We introduce the action of the torus $(\mathbb{C}^*)^n$ on the variety $X(\mathcal{T})$ in the same way as on the variety $\mathcal{G}$. Let $\sigma_t = (t_1, t_2, \ldots, t_n) \in (\mathbb{C}^*)^n$ act on coordinates in the following way:

$$\sigma_t(x_{i,j}) = t_i t_j x_{i,j}.$$

The action is well defined on the quotient ring $\mathbb{C}[X(\mathcal{T})]$, because generators of the ideal consist of polynomials which belong to the same grading. We say that an element $f \in \mathbb{C}[X(\mathcal{T})]$ belongs to a grading $\lambda \in \mathbb{Z}^n$ if:

$$\sigma_t(f) = t^{\lambda} f.$$

Note that this action of the torus is not faithful.

The grading of the algebra can be expressed in the combinatorial language of toric varieties. Let $\pi\colon \mathbb{Z}^{2n-3} \to \mathbb{Z}^n$ be the projection from the space spanned by the edges of the tree to the space spanned by the edges which are incident to leaves. In other words, π forgets about inner coordinates of the tree.

Recall that elements of $S(\mathcal{T})$ may be treated as monomials in $\mathbb{C}[X_{\mathrm{aff}}(\mathcal{T})]$, since $\mathbb{C}[X_{\mathrm{aff}}(\mathcal{T})]$ is isomorphic to $\mathbb{C}[S(\mathcal{T})]$.

Lemma 2. *An element $a \in S(\mathcal{T})$, treated as monomial in $\mathbb{C}[S(\mathcal{T})]$, belongs to the grading $\pi(a)$.*

Informally, it must be shown that the grading of the semigroup is determined by the values corresponding to the edges which are incident to leaves. To a grading $\lambda \in \mathbb{Z}^n$ belong elements of the semigroup having value λ_i on the edge incident to the ith leaf.

2.4. The degeneration of the Grassmannian to the variety described by a tree

A proof of the following theorem (stated in a much more general situation) may be found in ([6], Theorems 5.2, 5.3, 10.6). We present a simple proof for convenience of the reader (for a description of this idea, see [4, Chapter 15.8]).

Theorem 1. *The Grassmannian $\mathcal{G}$ degenerates to $X(\mathcal{T})$ for every tree $\mathcal{T}$. That is, there exists a variety $V \subseteq \mathbb{P}_{\mathbb{C}}^{n(n-1)/2-1} \times \mathbb{C}$ such that*

$$V_t \cong X(\mathcal{T}) \text{ for } t = 0,$$
$$V_t \cong \mathcal{G} \text{ for } t \neq 0,$$

where V_t is the fiber over $t \in \mathbb{C}$ under the projection to $\mathbb{C}$.

Proof. Let us recall that $d(i,j)$ is the length of $w_{i,j}$. Consider the following action of the torus $\mathbb{C}^*$ on $\mathbb{P}_{\mathbb{C}}^{n(n-1)/2-1}$:

$$t(x_{i,j}) = t^{-d(i,j)} x_{i,j}.$$

We define on $\mathbb{P}_{\mathbb{C}}^{n(n-1)/2-1} \times \mathbb{C}^*$ a variety V' given by the equations:

$$t(x_{i,j})t(x_{k,l}) - t(x_{i,k})t(x_{j,l}) + t(x_{i,l})t(x_{j,k}) = 0 \text{ for } 1 \leq i < j < k < l \leq n. \tag{1}$$

Using the isomorphism $V' \cong \mathcal{G} \times \mathbb{C}^*$ induced by the map $(x_{i,j}, t) \to (t(x_{i,j}), t)$, we get that $V'_t \cong \mathcal{G}$, where V'_t is the fiber over $t \in \mathbb{C}^*$ of the projection from V' to $\mathbb{C}^*$.

We would like to extend V' to $\mathbb{P}_{\mathbb{C}}^{n(n-1)/2-1} \times \mathbb{C}$. Let $V \subset \mathbb{P}_{\mathbb{C}}^{n(n-1)/2-1} \times \mathbb{C}$ be given by the following equations:

$$x_{i,j}x_{k,l} - x_{i,k}x_{j,l} + t^{d(i,k)+d(j,l)-d(i,l)-d(j,k)}x_{i,l}x_{j,k} = 0, \text{ if } w_{i,j} \text{ intersects } w_{k,l},$$
$$t^{d(i,l)+d(j,k)-d(i,j)-d(k,l)}x_{i,j}x_{k,l} - x_{i,k}x_{j,l} + x_{i,l}x_{j,k} = 0, \text{ if } w_{i,l} \text{ intersects } w_{j,k},$$

for any $1 \leq i < j < k < l \leq n$. To obtain these equations we multiplied (1) by the required power of t. The equations are well defined on $\mathbb{P}_{\mathbb{C}}^{n(n-1)/2-1} \times \mathbb{C}$ by Corollary 1.

We see that $V = V'$ for $t \neq 0$. Since, again by Corollary 1, the exponents of the powers of t in the above equations are positive, we have that $V_0 \cong X(\mathcal{T})$. □

The following proposition holds.

Proposition 3. *The Poincare-Hilbert series for $\mathcal{G}$ and $X(\mathcal{T})$, with respect to the action of the torus $(\mathbb{C}^*)^n$ described above, are the same.*

Let π_1 and π_2 be the projections from V to $\mathbb{P}_{\mathbb{C}}^{n(n-1)/2-1}$ and $\mathbb{C}$ respectively. Then, using the fact that the action of the torus on $\mathcal{G}$ and $X(\mathcal{T})$ extends to an action on the whole family V, the proposition follows from the Grauert theorem (see [7, Theorem III.9.9]) for the zeroth derived functor applied to the sheaf $\pi_1^*\mathcal{O}(m)$ for $m \in \mathbb{Z}_{>0}$ and the map π_2. Note that the dimensions of the eigenspaces of the action of the torus on $\pi_{2*}\pi_1^*\mathcal{O}(m)$ are equal to the sought-for dimensions. The assumptions of the Grauert theorem are satisfied, since a surjective projection from an irreducible variety to $\mathbb{C}$ is always flat. See also the proof of [2, Proposition 2.35].

Corollary 2. *The Poincare-Hilbert series for $X(\mathcal{T})$ is independent of the choice of a 3-valent tree $\mathcal{T}$ and is symmetric with respect to the variables $z_1, \ldots, z_n$.*

Proof. The independence follows straightforwardly from the proposition above. The series is symmetric, since it does not change if edges of $\mathcal{T}$ are relabeled. □

3. The methods of calculating the Poincare-Hilbert series

3.1. The semigroup of the variety described by a tree

We consider only paths which are the shortest paths between leaves, that is paths of the form $w_{i,j}$ for leaves $i < j$.

First, we introduce the notion of an "ordered intersection of paths" which will be used to single out a particular type of decomposition into sums of paths of $S(\mathcal{T})$ elements. In this section all pairs are ordered, that is for (i,j) it holds that $i<j$. We compare two ordered pairs lexicographically. The ordering on pairs induces the ordering on paths:

$$w_{i,j} < w_{k,l} \text{ if and only if } (i,j) < (k,l), \text{ where } i<j \text{ and } k<l.$$

Subsequently, it induces the lexicographic ordering on ordered pairs of paths. For paths $w_1 < w_2$ and $w_1' < w_2'$ we say that

$$(w_1,w_2) < (w_1',w_2') \text{ if and only if } w_1 < w_1' \text{ or both } w_1 = w_1' \text{ and } w_2 < w_2'.$$

We introduce a notion of duality. Consider an ordered pair of intersecting paths $(w_{a,b}, w_{c,d})$ without common endpoints, where $a<b$, $c<d$ and $w_{a,b} < w_{c,d}$. We can divide four leaves a,b,c,d into two pairs in exactly three ways. For each division we connect each pair of leaves by a path. Lemma 1 implies that in two out of three cases, the paths intersect (exactly for a pair $(w_{a,b}, w_{c,d})$ and for some ordered pair $(w_{a',b'}, w_{c',d'})$), and in one case they do not. We call the pair of paths $(w_{a',b'}, w_{c',d'})$ the *dual to the pair* $(w_{a,b}, w_{c,d})$. The dual pair is constructed from a pair of paths by exchanging two endpoints in such a way that the paths still intersect. Note that $w_{a,b} + w_{c,d} = w_{a',b'} + w_{c',d'}$ (see Figure 1).

In other words, the dual to our pair $(w_{a,b}, w_{c,d})$ is the ordered pair of intersecting paths $(w_{a',b'}, w_{c',d'})$, where a',b',c',d' are indices satisfying $a'<b'$, $c'<d'$, $w_{a',b'} < w_{c',d'}$ and $\{a',b',c',d'\} = \{a,b,c,d\}$. Observe that the dual pair is uniquely determined and the notion of duality is symmetric.

Definition 1. *Let us consider two intersecting paths $w_1 \in \mathbb{Z}^{2n-3}$ and $w_2 \in \mathbb{Z}^{2n-3}$ without common endpoints. Let (w_1', w_2') be the dual pair to (w_1,w_2). We say that w_1 intersects w_2 in an* ordered way *if $(w_1,w_2) < (w_1',w_2')$. Otherwise we say that they intersect in an* unordered way*.*

It is clear from the definition that the paths w_1, w_2 intersect in an ordered way if and only if the paths w_1', w_2' which are dual to (w_1,w_2) intersect in an unordered way. In the case when two paths have the same endpoint, which implies that they intersect, we say that the paths intersect in an ordered way.

Proposition 4. *Each $x \in S(\mathcal{T}) \subset \mathbb{Z}^{2n-3}$ decomposes into a sum of paths $w_{i,j}$ such that no two paths in this decomposition intersect in an unordered way.*

Proof. Let us choose the smallest lexicographical decomposition of x into a sum of paths – we treat a sum of paths as the ordered sequence of summands and compare lexicographically. We order paths in the way stated at the beginning of the section.

Suppose that two paths w and w' in this decomposition intersect in an unordered way. Let (v,v') be the dual pair (constructed from w,w' by replacing endpoints). After replacing the paths w and w' by v and v' in the decomposition of x, we still obtain a decomposition of x, since $w+w' = v+v'$. This new decomposition is lexicographically smaller, since the definition of an unordered intersection implies that $(v,v') < (w,w')$. $\square$

Proposition 5. *Let $x \in S(\mathcal{T}) \subset \mathbb{Z}^{2n-3}$. Then x has a unique decomposition into a sum of paths $w_{i,j}$ such that no two paths in the decomposition intersect in an unordered way.*

Proof. If $n = 3$, no pairs of paths intersect in an unordered way. Let

$$x = [x_1, x_2, x_3] = a_{1,2}w_{1,2} + a_{1,3}w_{1,3} + a_{2,3}w_{2,3},$$

where x_i is the value on the ith edge. Then $a_{1,2}, a_{2,3}, a_{1,3}$ are unique solutions of the system of three equations with three variables that is

$$a_{1,2} = \frac{1}{2}(x_1 + x_2 - x_3) \quad a_{1,3} = \frac{1}{2}(x_1 + x_3 - x_2) \quad a_{2,3} = \frac{1}{2}(x_2 + x_3 - x_1).$$

Therefore the decomposition is unique.

Let us assume that the decomposition is unique for trees having $n - 1$ leaves. Let us choose a vertex v incident to two leaves l_1 and l_2, where $l_1 < l_2$. Let $\mathcal{T}'$ be the tree constructed from $\mathcal{T}$ by erasing l_1 and l_2. Notice that v is a leaf in $\mathcal{T}'$. Let p_n be a projection from the space $\mathbb{Z}^{2n-3}$ of edges of the tree $\mathcal{T}$ to the space $\mathbb{Z}^{2n-5}$ of edges of the tree $\mathcal{T}'$. Observe that if $x \in S(\mathcal{T})$ then $p_n(x) \in S(\mathcal{T}')$.

Let $x = \sum_{i,j} a_{i,j}w_{i,j}$ for $a_{i,j} \in \mathbb{Z}_{\geq 0}$ be a decomposition into paths in which every two paths intersect in an ordered way. Then $p_n(x) = \sum a_{i,j}p_n(w_{i,j})$. Note that $\{p_n(w_{i,j}) \text{ for } a_{i,j} \neq 0\}$ is a set of paths in $\mathcal{T}'$ in which every two paths intersect in an ordered way. By induction this decomposition of $p_n(x)$ is unique.

Consequently, the paths in the decomposition of x which are disjoint to l_1 and l_2 are uniquely determined. Paths passing through v (except those between l_1 and l_2) are nearly uniquely determined – we do not know only whether they end in l_1 or l_2.

Let x_{l_1}, x_{l_2}, x_v be the values of x on three edges coming out of v, where x_{l_1}, x_{l_2} are the values on the edges which are incident to l_1 and l_2 respectively. Then a_{l_1,l_2} is equal to $(x_{l_1} + x_{l_2} - x_v)/2$ similarly to the case $n = 3$. So the number of paths leaving from l_1 and not entering l_2 is $y_{l_1} := x_{l_1} - (x_{l_1} + x_{l_2} - x_v)/2$. Analogously, the number of paths from l_2 which do not enter l_1 is $y_{l_2} := x_{l_2} - (x_{l_1} + x_{l_2} - x_v)/2$. Clearly $y_{l_1} + y_{l_2} = x_v$.

Paths starting in l_1 must end in leaves with smaller (or equal) numbers than paths starting in l_2, since otherwise we would have a pair of paths which intersect in an unordered way. Therefore paths which pass through v are uniquely determined – under the lexicographic ordering the first y_{l_1} paths which end in v in the decomposition of $p_n(x)$ must be extended to paths which end in l_1 and the other y_{l_2} must be extended to paths which end in l_2. $\square$

The two propositions mentioned above imply:

Corollary 3. *Elements of $S(\mathcal{T})$ are in bijection with such sums of $w_{i,j}$ in which no two paths intersect in an unordered way.*

We have shown that the Poincare-Hilbert series is independent of the choice of the tree $\mathcal{T}$ (see Corollary 2). We consider trees $\mathcal{T}_{n+1}$ with $n + 1$ leaves, called *caterpillar trees*, of the following form:

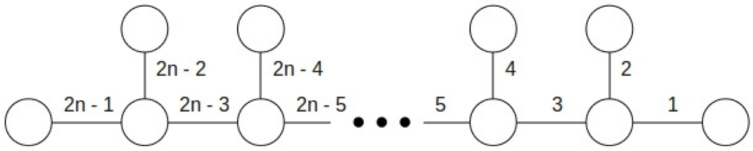

Figure 2: Tree $\mathcal{T}_{n+1}$

3.2. A combinatorial interpretation of dimensions of torus action's eigenspaces

Observe that $\dim(\mathbb{C}[X(\mathcal{T}_n)]_\lambda)$ is equal to the number of elements of $S(\mathcal{T}_n)$ which lie in a grading λ, since $\mathbb{C}[S(\mathcal{T}_n)] \cong \mathbb{C}[X(\mathcal{T}_n)]$ and the algebra $\mathbb{C}[S(\mathcal{T}_n)]$, considered as a group, is free.

Let $r_{i,j} \in \mathbb{Z}^n$ denote the vector with ones in the ith and jth coordinates ($i < j$) and zero everywhere else. We say that a vector $r_{i,j}$ (respectively pair (i,j)) *embraces* $r_{i',j'}$ (pair (i',j')), if $i < i' < j' < j$.

Theorem 2. *It holds that* $\dim(\mathbb{C}[X(\mathcal{T}_n)]_\lambda)$ *is equal to the number of decompositions of* $\lambda \in \mathbb{Z}^n$ *into a sum of vectors* $r_{i,j}$ *in which no term embraces any other term.*

For example, for $n = 4$ and $\lambda = [1,1,1,1]$ we have the following decompositions:

$$\lambda = [1,1,0,0] + [0,0,1,1] = [1,0,1,0] + [0,1,0,1] = [1,0,0,1] + [0,1,1,0].$$

The last decomposition is "invalid", because $[1,0,0,1]$ embraces $[0,1,1,0]$. This theorem implies that $\dim(\mathbb{C}[X(\mathcal{T}_4)])_{[1,1,1,1]} = 2$.

Proof. Firstly, note that in the tree $\mathcal{T}_n$ two paths $w_{i,j}$ and $w_{i',j'}$ intersect in an unordered way if and only if one pair embraces other one, that is, either (i,j) embraces (i',j') or (i',j') embraces (i,j).

Let $x \in S(\mathcal{T}_n)$ belong to the grading λ and be equal to $\sum_{i,j} a_{i,j} w_{i,j}$, where $a_{i,j} \in \mathbb{Z}_{\geq 0}$. Then

$$\lambda = \sum_{i,j} a_{i,j} r_{i,j}.$$

Corollary 3 shows that the elements of $S(\mathcal{T}_n)$ are in bijection with sums of paths, for which no path intersects another path in an unordered way.

The remarks above show that the elements of $S(\mathcal{T}_n)$ which lie in a grading λ are in bijection with decompositions of λ into sums of vectors $r_{i,j}$ such that no vector embraces any other. $\square$

3.3. Formulas for Poincare-Hilbert series

Let us define a function Multi from sequences of ordered pairs of integers (i,j) such that $1 \leq i < j \leq n$ to $\mathbb{Z}[z_1, \ldots, z_n]$.

$$\mathrm{Multi}((i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)) = \prod_{\text{over distinct pairs } (i_l,j_l)} z_{i_l} z_{j_l}.$$

For example $\mathrm{Multi}((1,2),(1,3),(1,2),(2,4)) = z_1z_2 \cdot z_1z_3 \cdot z_2z_4 = z_1^2z_2^2z_3z_4$.

We also define a function Sum from sequences of ordered pairs of integers (i,j) such that $1 \le i < j \le n$ to $\mathbb{Z}^n$.

$$\mathrm{Sum}((i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)) = \sum_{\text{over distinct pairs } (i_l,j_l)} r_{i_l,j_l}.$$

For example $\mathrm{Sum}((1,2),(1,3),(1,2),(2,4)) = r_{1,2} + r_{1,3} + r_{2,4} = [1,1,0,0] + [1,0,1,0] + [0,1,0,1] = [2,2,1,1]$.

It holds that

$$\mathrm{Multi}((i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)) = z^{\mathrm{Sum}((i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k))}. \tag{2}$$

Let $\mathrm{Exc} = \{((i,j),(i',j')) \mid 1 \le i < i' < j' < j \le n;\ i,j,i',j' \in \mathbb{Z}_{\ge 0}\}$. In other words, Exc is a set of 2-tuples of pairs such that the first pair embraces the second one. We introduce a natural lexicographic ordering on Exc. Let $e^1 = (i,j)$ and $e^2 = (i',j')$ for $e = ((i,j),(i',j'))$.

We define functions $\widetilde{\mathrm{Multi}}$ from sequences of elements of Exc to $\mathbb{Z}[z_1,\ldots,z_n]$ and $\widetilde{\mathrm{Sum}}$ from sequences of elements of Exc to $\mathbb{Z}^n$ in the following way:

$$\widetilde{\mathrm{Multi}}(e_1,\ldots,e_n) := \mathrm{Multi}(e_1^1,e_1^2,\ldots,e_k^1,e_k^2), \text{ where } e_1,\ldots,e_n \in \mathrm{Exc},$$
$$\widetilde{\mathrm{Sum}}(e_1,\ldots,e_n) := \mathrm{Sum}(e_1^1,e_1^2,\ldots,e_k^1,e_k^2), \text{ where } e_1,\ldots,e_n \in \mathrm{Exc}\,.$$

Theorem 3. *The Poincare-Hilbert series W_n for $X(\mathcal{T}_n)$ is equal to*

$$\frac{1}{\prod_{1\le i<j\le n}(1-z_iz_j)}\Big(1 + \sum_{k=1}^{n}(-1)^k \sum_{\substack{e_1<\ldots<e_k \\ e_1,\ldots,e_k\in \mathrm{Exc}}} \widetilde{\mathrm{Multi}}(e_1,\ldots,e_k)\Big).$$

Proof. Let Ω^λ denote the set of all decompositions of $\lambda \in \mathbb{Z}^n$ into sums of vectors $r_{i,j}$. Let A^λ denote the set of all decompositions for which no summand embraces other one. Let $\Omega^\lambda_{(a,b),(a',b')}$ denote the set of all decompositions in which both summands $r_{a,b}$ and $r_{a',b'}$ occur.

Theorem 2 implies that $\dim(\mathbb{C}[X(\mathcal{T}_n)]_\lambda) = |A^\lambda|$. By definition

$$A^\lambda = \Omega^\lambda \setminus \bigcup_{e\in \mathrm{Exc}} \Omega^\lambda_e.$$

By inclusion-exclusion formula we have that

$$|A^\lambda| = |\Omega^\lambda| - \sum_{k=1}^{n}(-1)^{k-1} \sum_{\substack{e_1<\ldots<e_k \\ e_1,\ldots,e_k\in \mathrm{Exc}}} |\bigcap_{l=1}^{k} \Omega^\lambda_{e_l}|. \tag{3}$$

Let

$$W(\Omega) = \sum_{\lambda\in\mathbb{Z}^n} |\Omega^\lambda| z^\lambda.$$

Note that $\bigcap_{l=1}^k \Omega_{e_l}^\lambda$ consists of exactly those decompositions which contain summands $r_{e_1^1}$, $r_{e_1^2}$, ..., $r_{e_k^1}$, $r_{e_k^2}$ (note that subsequent occurrences of the pairs should be omitted – each pair is considered at most once). The element, in the decomposition of which each of these summands is contained exactly ones, is equal to $\widetilde{\mathrm{Sum}}(e_1,\ldots,e_k)$. There is a natural bijection between decompositions of λ containing summands mentioned above and arbitrary decompositions of $\lambda - \widetilde{\mathrm{Sum}}(e_1,\ldots,e_k)$. Therefore

$$|\bigcap_{l=1}^k \Omega_{e_l}^\lambda| = |\Omega^{\lambda - \widetilde{\mathrm{Sum}}(e_1,\ldots,e_k)}|.$$

For fixed $e_1,\ldots,e_k \in \mathrm{Exc}$ we have that

$$\begin{aligned}
\sum_{\lambda\in\mathbb{Z}^n} |\bigcap_{l=1}^k \Omega_{e_l}^\lambda| z^\lambda &= \sum_{\lambda\in\mathbb{Z}^n} |\Omega^{\lambda - \widetilde{\mathrm{Sum}}(e_1,\ldots,e_k)}| z^\lambda \\
&\quad \text{take } \lambda' = \lambda - \widetilde{\mathrm{Sum}}(e_1,\ldots,e_k) \\
&= \sum_{\lambda'\in\mathbb{Z}^n} |\Omega^{\lambda'}| z^{\lambda'} z^{\widetilde{\mathrm{Sum}}(e_1,\ldots,e_k)} \\
&\overset{(2)}{=} \sum_{\lambda'\in\mathbb{Z}^n} |\Omega^{\lambda'}| z^{\lambda'} \widetilde{\mathrm{Multi}}(e_1,\ldots,e_k) \\
&= W(\Omega)\widetilde{\mathrm{Multi}}(e_1,\ldots,e_k).
\end{aligned}$$

Let us multiply both sides of equality (3) by z^λ and sum over $\lambda \in \mathbb{Z}^n$. We obtain:

$$W_n = W(\Omega) + \sum_{k=1}^n (-1)^k \sum_{\substack{e_1<\ldots<e_k \\ e_1,\ldots,e_k\in\mathrm{Exc}}} W(\Omega)\widetilde{\mathrm{Multi}}(e_1,\ldots,e_l).$$

The conclusion of the theorem follows from the following lemma:

Lemma 3. *It holds that*

$$W(\Omega) = \frac{1}{\prod_{1\le i<j\le n}(1 - z_i z_j)}.$$

Proof. Observe that

$$\frac{1}{\prod_{1\le i<j\le n}(1 - z_i z_j)} = \prod_{1\le i<j\le n} (1 + z_i z_j + (z_i z_j)^2 + \ldots).$$

After expanding this product, we see that the coefficient of the z^λ term is equal to the number of decompositions of λ into the sum of $r_{i,j}$, which proves the lemma. □

□

Note that if we define Exc as a set of pairs of paths intersecting in an unordered way, then the theorem will be true for an arbitrary tree with an analogous proof.

3.4. A recursive formula for Poincare-Hilbert series

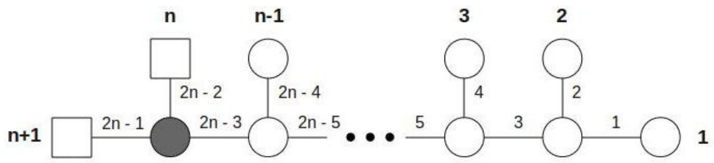

Figure 3: Tree $\mathcal{T}_{n+1}$

We try to calculate the semigroup $S(\mathcal{T}_{n+1})$ from $S(\mathcal{T}_n)$ recursively. The tree $\mathcal{T}_{n+1}$ can be constructed from $\mathcal{T}_n$ by adding two vertices (squares) and corresponding incident edges to the last leaf of $\mathcal{T}_n$ (filled circle). The last leaf of $\mathcal{T}_n$ becomes an interior vertex.

Let us denote by p_{n+1} the projection $\mathbb{Z}^{2n-1} \to \mathbb{Z}^{2n-3}$

$$p_{n+1}(s_1, \dots, s_{2n-3}, s_{2n-2}, s_{2n-1}) = (s_1, \dots, s_{2n-3}), \text{ where } s_i \in \mathbb{Z}.$$

We treat p_{n+1} as the restriction of the lattice $\mathbb{Z}^{2n-1}$ spanned by edges of $\mathcal{T}_{n+1}$ to the lattice $\mathbb{Z}^{2n-3}$ spanned by edges of $\mathcal{T}_n$, in which the values of the last two edges (those with biggest numbers – see Figure (3)) of the tree $\mathcal{T}_{n+1}$ in the lattice $\mathbb{Z}^{2n-1}$ are "forgotten". Recall that $S(\mathcal{T}_{n+1}) \subset \mathbb{Z}_{2n-1}$ and $S(\mathcal{T}_n) \subset \mathbb{Z}_{2n-3}$.

Let $w_{i,j} \in \mathbb{Z}^{2n-1}$ be the shortest path between ith and jth leaves in $\mathcal{T}_{n+1}$ and let $\widetilde{w}_{i,j} \in \mathbb{Z}^{2n-3}$ be the shortest path between ith and jth leaves in $\mathcal{T}_n$.

Under the restriction p_{n+1},

- the path between the last two leaves of $\mathcal{T}_{n+1}$ "becomes empty", that is
$$p_{n+1}(w_{n,n+1}) = 0.$$
- other paths which start in the last two leaves shrink by one and subsequently they start at the last leaf of $\mathcal{T}_n$, that is
$$p_{n+1}(w_{i,n}) = \widetilde{w}_{i,n} \text{ for } i \neq n+1,$$
$$p_{n+1}(w_{i,n+1}) = \widetilde{w}_{i,n} \text{ for } i \neq n.$$
(remember that the nth leaf of $\mathcal{T}_n$ is not the nth leaf of $\mathcal{T}_{n+1}$)
- all other paths do not change
$$p_{n+1}(w_{i,j}) = \widetilde{w}_{i,j} \text{ for } i, j \neq n, n+1.$$

Therefore, p_{n+1} maps paths in $S(\mathcal{T}_{n+1})$ to paths in $S(\mathcal{T}_n)$. Every path in $S(\mathcal{T}_n)$ is in the image, so p_{n+1} restricts to a surjective homomorphism of semigroups from $S(\mathcal{T}_{n+1})$ to $S(\mathcal{T}_n)$.

Lemma 4. *Let $s = (s_1, \dots, s_{2n-3}, s_{2n-2}, s_{2n-1}) \in S(\mathcal{T}_{n+1})$. Then $s_{2n-2} = \beta + k$ and $s_{2n-1} = \gamma + k$ for some $k \in \mathbb{N}$ and $\beta, \gamma \in \mathbb{Z}_{\geq 0}$ such that $\beta + \gamma = s_{2n-3}$.*

The statement of the lemma is equivalent to claiming that the sum of the values on the last two edges in $\mathcal{T}_{n+1}$ must be greater or equal than the value on the preceding edge and differs from it by paths between last two leaves.

Proof. Let $s = \sum_{1 \le i < j \le n+1} a_{i,j} w_{i,j}$, where $a_{i,j} \in \mathbb{Z}_{\ge 0}$. Let $\beta = \sum_{i<n} a_{i,n}$ and $\gamma = \sum_{i<n} a_{i,n+1}$ be the number of paths different from the path between last two leaves and passing through the edges $2n-2$ and $2n-1$ respectively.

Obviously $s_{2n-2} = \beta + a_{n,n+1}$ and $s_{2n-1} = \gamma + a_{n,n+1}$. Each path crossing the edge $2n-3$ crosses either the edge $2n-1$ or the edge $2n-2$, so $\gamma + \beta = s_{2n-3}$. Take $k = a_{n,n+1}$. □

Lemma 5. *It holds that*

$$S(\mathcal{T}_{n+1}) = \Big\{ (s_1, s_2, \ldots, s_{2n-3}, \beta + k, \gamma + k) \;\Big|\; (s_1, s_2, \ldots, s_{2n-3}) \in S(\mathcal{T}_n), \\ k \in \mathbb{Z}_{\ge 0} \text{ and } \beta + \gamma = s_{2n-3} \Big\}.$$

Proof. The last lemma implies that the elements of $S(\mathcal{T}_{n+1})$ must be of this form. We now prove that every element of this form belongs to the semigroup.

Let $s = (s_1, s_2, \ldots, s_{2n-3}, \beta + k, \gamma + k)$ for $k \in \mathbb{Z}_{\ge 0}$ and $\beta + \gamma = s_{2n-3}$. Let

$$(s_1, s_2, \ldots, s_{2n-3}) = \sum_{(i,j) \in P} \widetilde{w}_{i,j},$$

where P is a multiset (the structure in which one element may occur many times) of ordered pairs of leaves of $\mathcal{T}_n$.

It holds that $s_{2n-3} = |\{(i,n) \in P\}|$, that is s_{2n-3} equals the number of elements of the form (i,n) in P counted with multiplicity. We divide this multiset $\{(i,n) \in P\}$ arbitrarily into two multisets P', P'' with β and γ elements respectively.

We see that

$$s = \sum_{(i,j) \in P;\; i,j < n} w_{i,j} + \sum_{(i,n) \in P'} w_{i,n} + \sum_{(i,n) \in P''} w_{i,n+1} + k w_{n,n+1}.$$

Therefore $s \in S(\mathcal{T}_{n+1})$.

□

Let W_n be the Poincare-Hilbert series for $X(\mathcal{T}_n)$. Let us recall that $\dim(\mathbb{C}[X(\mathcal{T}_n)]_\lambda)$ is equal to the number of elements in $S(\mathcal{T}_n)$ which belong to the grading λ, because $\mathbb{C}[S(\mathcal{T}_n)] \cong \mathbb{C}[X(\mathcal{T}_n)]$ and the algebra $\mathbb{C}[S(\mathcal{T}_n)]$, considered as a group, is free.

Theorem 4. *Let $W_n = \sum_{i=0}^{\infty} w_i z_n^i$, where $w_i \in \mathbb{Z}[\![z_1, \ldots, z_{n-1}]\!]$. Then for $n \ge 3$:*

$$W_{n+1} = \left(\sum_{i=0}^{\infty} w_i \cdot (\sum_{l=0}^{i} z_n^{i-l} z_{n+1}^l) \right) \cdot \frac{1}{1 - z_n z_{n+1}}.$$

Proof. Lemma 2 implies that the grading of an element of a semigroup is the image of that element's projection to the space of leaves. For each element $s = (s_1, s_2, \ldots, s_{2n-3}) \in S(\mathcal{T}_n)$, we must understand to which grading the elements of $S(\mathcal{T}_{n+1})$ restricting to s belong.

Let $(\alpha_1, \ldots, \alpha_n)$ be the grading of the element $(s_1, \ldots, s_{2n-3}) \in S(\mathcal{T}_n)$. Then the element $(s_1, s_2, \ldots, s_{2n-3}, \beta + k, \gamma + k)$, where $\beta + \gamma = s_{2n-3}$ belongs to the grading $(\alpha_1, \ldots, \alpha_{n-1}, \beta + k, \gamma + k)$.

Therefore we get W_{n+1} from W_n by changing z_n^i into the sum (see Lemma 5):

$$\sum_{k \geq 0, \beta+\gamma=i} z_n^{\beta+k} z_{n+1}^{\gamma+k}.$$

The theorem now follows from the equality:

$$\begin{aligned} \sum_{k \geq 0, \beta+\gamma=i} z_n^{\beta+k} z_{n+1}^{\gamma+k} &= \left(\sum_{\beta+\gamma=i} z_n^{\beta} z_{n+1}^{\gamma} \right) \cdot \left(\sum_{k \geq 0} (z_n z_{n+1})^k \right) \\ &= \left(\sum_{l=0}^{i} z_n^{i-l} z_{n+1}^{l} \right) \cdot \frac{1}{1 - z_n z_{n+1}} \end{aligned}$$

□

It is easy to see that the formula is also true for $n = 2$.

3.5. Other formulas for Poincare-Hilbert series

These recursive formulas describe infinite objects (series), which is problematic, because they do not allow "mechanical" calculations. Proposition 1 implies that our series W_n can be presented in the following form:

$$\frac{F_n(z_1, \ldots, z_n)}{\prod_{1 \leq i < j \leq n} (1 - z_i z_j)}, \text{ where } F_n \in \mathbb{Z}[z_1, \ldots, z_n].$$

For performing calculations, the recursive formula for polynomials F_n would be more valuable. Before we formulate it, let us define a few notions. Let:

$$F_n := \sum_{i=0}^{d} f_i z_n^i, \text{ where } f_i \in \mathbb{Z}[z_1, \ldots, z_{n-1}] \text{ and } F_{n+1} = \sum_{i=0} \widetilde{f}_i z_{n+1}^i.$$

Let h_i be a sum of all monomials of degree i and let σ_i be the ith elementary symmetric polynomial (both of $n-1$ variables $z_1, \ldots, z_{n-1}$). Let:

$$H_{s,l} := \sum_{r=0}^{l} h_{s-r} \cdot \sigma_r \cdot (-1)^r.$$

Lemma 6. *The following formulas hold:*

$$\widetilde{f}_t = \sum_{i=0}^{d} f_i a_{t-i,i}, \text{ where:}$$

$$a_{k,l} = \sum_{\beta} z_n^{\beta} \sum_{\alpha=0}^{k+l} (-1)^{\alpha} \sigma_{\alpha} H_{k+\beta-\alpha,\beta}.$$

As the proof of the lemma is quite technical, we omit it in the paper. See [14] for the implementation of the formula in Sage.

Remark 2. *We can verify the formulas for the Poincare-Hilbert series in the case of $n = 5$ in the following way. Let V be a projective plane $\mathbb{CP}^2$ blown-up in four general points p_1, p_2, p_3 and p_4. By $E_{0,i}$, for $1 \leq i \leq 4$, we denote the exceptional divisor corresponding to p_i and by $E_{i,j}$, for $1 \leq i < j \leq 4$, we denote the lifting to V of the line which passes through p_i and p_j. For $1 \leq i < j \leq 4$, let us introduce the notion $\overline{E_{i,j}} := E_{k,l}$, where $\{k,l\} = \{1,2,3,4\} \backslash \{i,j\}$ and $\overline{E_{0,i}} = E_{0,i}$, where $1 \leq i \leq 4$. It is a classical result (see the introduction to [10]) that:*

$$\mathbb{C}[\mathcal{G}_5] = \bigoplus_{\lambda \in \mathbb{Z}^5} \mathbb{C}[\mathcal{G}_5]_{\lambda} \cong \bigoplus_{D \in \mathrm{Pic}(V)} H^0(V, \mathcal{O}(D)).$$

This is an isomorphism of rings with grading in $\mathbb{Z}^5$. The divisor $D = \sum a_{i,j} \overline{E_{i,j}} \in \mathrm{Pic}(V)$ corresponds to the grading $\lambda(D) = \sum a_{i,j} r_{i,j}$, where $r_{i,j} \in \mathbb{Z}^5$ has ones on the ith and jth coordinates and zeroes on the other coordinates. Take a divisor $D = \sum_{0 \leq i < j \leq 4} a_{i,j} \overline{E_{i,j}}$. Then this classical result implies that the coefficient of the monomial term $\prod (z_i z_j)^{a_{i,j}}$ in the Poincare-Hilbert series for $\mathcal{G}_5$ for the action of the torus described in preliminaries, is equal to $h^0(\mathcal{O}(D))$. One can verify the formulas obtained in this paper by calculating coefficients of the Poincare-Hilbert series corresponding to the divisors whose nonzero cohomologies vanish. For such divisors D, we have that $h^0(\mathcal{O}(D))$ is equal to $\chi(\mathcal{O}(D))$, which can be calculated using the Riemann-Roch theorem.

A. Examples of the Poincare-Hilbert series for small dimensions

The Poincare-Hilbert series for small n:

- for $n = 2$
$$\frac{1}{1 - z_1 z_2}$$
- for $n = 3$
$$\frac{1}{(1 - z_1 z_2)(1 - z_2 z_3)(1 - z_3 z_1)}$$
- for $n = 4$
$$\frac{1 - z_1 z_2 z_3 z_4}{\prod_{1 \leq i < j \leq 4} (1 - z_i z_j)}$$

- for $n = 5$

$$(-z_0^2 z_1^2 z_2^2 z_3^2 z_4^2 + z_0^2 z_1 z_2 z_3 z_4 + z_0 z_1^2 z_2 z_3 z_4 + z_0 z_1 z_2^2 z_3 z_4 + z_0 z_1 z_2 z_3^2 z_4 + z_0 z_1 z_2 z_3 z_4^2 - z_0 z_1 z_2 z_3 - z_0 z_1 z_2 z_4 - z_0 z_1 z_3 z_4 - z_0 z_2 z_3 z_4 - z_1 z_2 z_3 z_4 + 1) / \prod_{1 \le i < j \le 5} (1 - z_i z_j)$$

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